$\boxed{1}$ $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

- $c A(x) \longleftrightarrow (c a_n)_{n \geq 0}$
- $x A(x) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots \longleftrightarrow (0, a_0, a_1, a_2, \ldots)$
- $A(x) + B(x) \longleftrightarrow (a_n + b_n)_{n \geq 0}$
- $A(x) B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots \longleftrightarrow (c_n)_{n \geq 0}$

where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$

$\boxed{2}$ $S = 1 + x + x^2 + \cdots$ $\quad S - xS = S(1-x) = 1 \rightarrow S = \frac{1}{1-x}$

$xS = \quad x + x^2 + \cdots$

$\boxed{3}$ $\frac{A(x)}{1-x} = A(x)(1 + x + x^2 + \cdots) = (a_0 + a_1 x + a_2 x^2 + \cdots)(1 + x + x^2 + \cdots)$

$= a_0 + (a_0 + a_1) x + (a_0 + a_1 + a_2) x^2 + \cdots$

$\longleftrightarrow \left(\sum_{k=0}^{n} a_k \right)_{n \geq 0}$

$\boxed{4}$

- $(1, 0, 1, 0, \ldots) \longleftrightarrow 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$
- $(1, 2, 4, 8, 16, \ldots) \longleftrightarrow 1 + 2x + 4x^2 + 8x^3 + \cdots = 1 + 2x + (2x)^2 + (2x)^3 + \cdots = \frac{1}{1-2x}$
- $(1, 2, 3, 4, 5, \ldots) \longleftrightarrow 1 + 2x + 3x^2 + 4x^3 + \cdots$

$= (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)$

$= \frac{1}{(1-x)^2}$

$\boxed{5}$ $f_n = n^{th}$ Fibonacci #. $\quad F(x) = f_0 + f_1 x + f_2 x^2 + \dots$

- $(0, f_0, f_1, f_2, \dots) \longleftrightarrow f_0 x + f_1 x^2 + f_2 x^3 + \dots = xF(x) = G(x)$
- $(0, 0, f_0, f_1, \dots) \longleftrightarrow f_0 x^2 + f_1 x^3 + f_2 x^4 + \dots = x^2 F(x) = H(x)$
- $F(x) - G(x) - H(x) = f_0 + (f_1 - f_0)x + (f_2 - f_1 - f_0)x^2 + (f_3 - f_2 - f_1)x^3 + \dots$

$$= f_0 + (f_1 - f_0)x = x$$

- $x = F(x) - G(x) - H(x) = F(x) - xF(x) - x^2F(x) = F(x)(1 - x - x^2)$

$$\Rightarrow F(x) = \frac{x}{1 - x - x^2}$$

$\boxed{6}$ $F(x) = \dfrac{-x}{x^2 + x - 1} = \dfrac{-x}{(x-a)(x-b)}$ where $a = \dfrac{-1+\sqrt{5}}{2}$, $b = \dfrac{-1-\sqrt{5}}{2}$

$$= \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\frac{1}{x-a} + \left(\frac{-1-\sqrt{5}}{2\sqrt{5}}\right)\frac{1}{x-b}$$

$\boxed{7}$ $\cancel{F(x) = \frac{\frac{1}{2}(\sqrt{5}+3)}{-a}\left(\frac{1}{1-\frac{x}{a}}\right) + \frac{\frac{1}{2}(-\sqrt{5}-5)}{-b}\left(\frac{1}{1-\frac{x}{b}}\right) = (-2-\sqrt{5})\left(\frac{1}{1-\frac{x}{a}}\right) + (-\sqrt{5})\left(\frac{1}{1-\frac{x}{b}}\right)}$

$\cancel{=}$

$$F(x) = \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{-1}{a}\right)\left(\frac{1}{1-\frac{x}{a}}\right) + \left(\frac{-1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{-1}{b}\right)\left(\frac{1}{1-\frac{x}{b}}\right) = \frac{1}{\sqrt{5}}\frac{1}{1-\frac{x}{a}} + \frac{-1}{\sqrt{5}}\frac{1}{1-\frac{x}{b}}$$

$$= \frac{1}{\sqrt{5}}\left(1 + \frac{x}{a} + \left(\frac{x}{a}\right)^2 + \dots\right) - \frac{1}{\sqrt{5}}\left(1 + \frac{x}{b} + \left(\frac{x}{b}\right)^2 + \dots\right)$$

$$\Rightarrow f_0 = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0$$

$$f_1 = \frac{1}{a\sqrt{5}} - \frac{1}{b\sqrt{5}} = 1$$

$$\vdots$$

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1}{a}\right)^n - \left(\frac{1}{b}\right)^n\right) = \boxed{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}$$

$\boxed{8}$ $a_{n+1} = 2a_n + n, \quad a_0 = 1.$

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$2x\,A(x) = 0 + 2a_0 x + 2a_1 x^2 + \cdots$$

$$B(x) = 0 + 0x + 1x^2 + 2x^3 + \cdots = x^2(1 + 2x + 3x^2 + \cdots) = \frac{x^2}{(1-x)^2}$$

$$A(x) - 2xA(x) - B(x) = a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1 - 1)x^2 + (a_3 - 2a_2 - 2)x^3 + \cdots$$

$$= 1$$

$$A(x) \cdot (1-2x) = 1 + B(x) = 1 + \frac{x^2}{(1-x)^2} = \frac{(1-x)^2 + x^2}{(1-x)^2} \Rightarrow A(x) = \frac{2x^2 - 2x + 1}{(1-2x)(1-x)^2}$$

$$A(x) = \frac{2}{1-2x} - \frac{0}{1-x} - \frac{1}{(1-x)^2} = 2\left(1 + 2x + (2x)^2 + (2x)^3 + \cdots\right) - \left(1 + 2x + 3x^2 + 4x^3 + \cdots\right)$$

$$\Rightarrow a_0 = 2 - 1 = 1$$

$$a_1 = 2 \cdot 2 - 2 = 2$$

$$a_2 = 2 \cdot 2^2 - 3 = 5$$

$$a_3 = 2 \cdot 2^3 - 4 = 12$$

$$\boxed{a_n = 2 \cdot 2^n - (n+1)}$$

$\boxed{9}$ • $A(1) = \#$ sides on the die

• k^{th} coefficient of $A(x)B(x) = a_0 b_k + a_1 b_{k-1} + \dots + a_{k-1} b_1 + a_k b_0$

$= \#(\text{die } A = 0, \text{ die } B = k) + \dots + \#(\text{die } A = k, \text{ die } B = 0)$

$= \#$ ways die $A +$ die $B = k$.

• $$c_k = \frac{\#\text{ ways sum} = k}{\#\text{ total outcomes}} = \frac{\#\text{ ways sum} = k}{A(1)\,B(1)}$$

$$c_0 + c_1 x + c_2 x + \dots = \frac{A(x)\,B(x)}{A(1)\,B(1)}$$

$\boxed{10}$ need to find a different factorization $(x+x^2+x^3+x^4+x^5+x^6)^2 = A(x)\,B(x)$.

$$x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$$

$$\|$$

$$(x + 2x^2 + 2x^3 + x^4)(x + x^3 + x^4 + x^5 + x^6 + x^8)$$