Generating Functions

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"A generating function is a clothesline on which we hang up a sequence of numbers for display."
–Herbert Wilf [1]

1 Introduction

Let's say we have a sequence of numbers $a_0, a_1, a_2, a_3, \dots$, which we will refer to as $(a_n)_0^\infty$. A useful tool for studying this series is its *generating function*:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Under some circumstances, we can treat this like an honest function, but sometimes this doesn't work, because the infinite sum does not actually converge. We won't worry about that too much, as we can still manipulate these infinite sums using the rules of algebra, and get useful results even when the sums *don't* add up to anything.

Let's see how to manipulate these:

Problem 1 Let $A(x)$ be the generating function of $(a_n)_0^\infty$, and $B(x)$ the generating function of $(b_n)_0^\infty$. Find the sequences corresponding to the following generating functions:

- $cA(x)$
- $xA(x)$
- $A(x) + B(x)$
- $A(x)B(x)$

Problem 2 Prove that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$, at least when both sides are actually defined ($|x| < 1$).

Problem 3 Now, if $A(x)$ is the generating function of $(a_n)_0^\infty$, find what sequence corresponds to the generating function of $\frac{A(x)}{1-x}$.

[*] Inspired by the beginning generatingfunctionology by Herbert Wilf and notes by Padraic Bartlett

[1] The opening sentence of the fantastic book generatingfunctionology, available for free at `https://www.math.upenn.edu/~wilf/DownldGF.html`

Problem 4 Find short expressions for the generating functions for the following sequences:

- $1, 0, 1, 0, 1, 0, \ldots$
- $1, 2, 4, 8, 16, \ldots$
- $1, 2, 3, 4, 5, \ldots$

2 Fibonacci

Let's take a specific, familiar case: the Fibonacci numbers. To recall, these are a sequence $(f_n)_0^\infty$ where $f_0 = 0$ and $f_1 = 1$, and the rest of the sequence is defined by $f_n = f_{n-1} + f_{n-2}$. This last equation is called a *recurrence relation*, because it allows you to calculate the numbers step-by-step from previous entries in the sequence, rather than being able to directly calculate, say, f_{10000} from scratch. We will try to use generating functions to find a formula for f_n that doesn't refer to any other Fibonacci numbers.

Problem 5 Let $F(x)$ be the generating function for the sequence $f_0, f_1, f_2, \ldots$.

- Can you find the generating function for $0, f_0, f_1, f_2, \ldots$ in terms of $F(x)$? Let's call this $G(x)$.
- Can you find the generating function for $0, 0, f_0, f_1, f_2, \ldots$ in terms of $F(x)$? Let's call this $H(x)$.
- Using the recurrence relation, calculate $F(x) - G(x) - H(x)$.
- Put this all together to find an equation that $F(x)$ satisfies.
- Now solve for $F(x)$, expressing it as a rational function (a fraction where both numerator and denominator are polynomials).

Problem 6 Now that we have a rational function expression for $F(x)$, let's find a closed-form expression for its coefficients. Let's use a strategy called Partial Fraction Decomposition. Say $F(x) = \frac{p(x)}{(x-a)(x-b)}$, where $p(x)$ is a linear polynomial, and a, b are constants. Break it up and express it as a sum $\frac{p(x)}{(x-a)(x-b)} = \frac{c}{x-a} + \frac{d}{x-b}$, where c, d are constants.

Problem 7 Now using material from the introduction, and the most recent expression for $F(x)$, calculate a formula for its coefficients, and thus for the Fibonacci numbers in general.

Problem 8 Repeat the method of recurrence, generating function, partial fraction decomposition, geometric series, to find a closed form for the following sequence:

$$a_{n+1} = 2a_n + n \, (n \geq 0, a_0 = 1)$$

Hint: When doing partial fraction decomposition, if the denominator is of the form $(x-a)^2(x-b)$, you may have to express your fraction as a sum of three fractions, $\frac{c}{(x-a)^2} + \frac{d}{x-a} + \frac{e}{x-b}$.

3 Dice

Ignoring geometrical constraints, a die is a device that allows you to choose randomly and with equal probability one from a finite set of sides, each of which has a positive integer written on it. A standard die is one where the n sides are labeled with the numbers $1, 2, 3, \ldots, n$, such as the 6-sided dice we play many board games with, or the dice with $n = 4, 8, 10, 12, 20$, which come up in Dungeons and Dragons. However, we can come up with dice that have any numbers on their sides, and can even repeat numbers, such as a 5-sided die with $1, 1, 2, 3, 5$ as its sides, where 1 comes up twice as often as any other number.

To describe a die with a generating function, let a_k be the number of times the number k appears as a side of the die, and then consider $a_0+a_1x+a_2x^2+\ldots$. This should end up a regular polynomial, because only finitely many of these terms will be nonzero. For instance, the generating function of a regular 6-sided die will be $x + x^2 + x^3 + x^4 + x^5 + x^6$, and the generating function of our weird 5-sided die will be $2x + x^2 + x^3 + x^5$.

Problem 9 Let $A(x), B(x)$ be the generating functions of two dice.

- What is the significance of the number $A(1)$?
- Using formulas established earlier, show that the kth coefficient of $A(x)B(x)$ is the number of ways to roll a k as the sum of the two dice.
- Can you find a generating function for the sequence $c_0, c_1, c_2, \ldots$ where c_k is the *probability* that the sum of the two dice is k?

Problem 10 Using the techniques of generating functions, can you find two non-standard 6-sided dice (with only positive integer sides) such that their sum has the same distribution as the sum of two standard dice? That is, for any integer k, the number of ways that the sum of the two nonstandard dice comes up as k equals the number of ways that the sum of the two standard dice comes up k.

Hint. This problem relies on factoring polynomials.

4 Coins

There's a famous elementary problem that goes something like this: How many ways can you assemble pennies, nickels, dimes, quarters and half-dollars to reach a total of, say, 50¢? Most ways of solving this involve awkward brute-force approaches, and won't tell you anything about how to change your answer if instead you want to know how many ways to make change for 51¢, or $1.05. With generating functions, we can solve the problem in a general way.

Let $p_0, p_1, p_2, \ldots$ be such that p_k is the number of ways to make change for k cents with only pennies. Let n_k be the number of ways to make change for k cents with only nickels, d_k the number of ways using only dimes, q_k with only quarters, and h_k with only half-dollars.

Problem 11

- Let $p(x)$ be the generating function for $p_0, p_1, p_2, \ldots$. Express $p(x)$ as a rational function.
- Modify your expression for $p(x)$ to give an expression for the generating functions $n(x), d(x), q(x), h(x)$ of the other sequences.

Now let $N(x)$ be the generating function for the sequence $N_0, N_1, N_2, \ldots$ where N_k is the number of ways to make change for k cents using pennies *and* nickels. Similarly, let $D(x)$ be the generating function for the sequence using pennies, nickels, and dimes, let $Q(x)$ use pennies, nickels, dimes, and quarters, and let $H(x)$ use all the coins.

Problem 12

- Find a rational function expression for $N(x)$.
- Using your expression for $N(x)$, find a rational function expression for $D(x)$, and then $Q(x)$, and then $H(x)$.
- Using these generating functions, find recurrence relations for these sequences. Your recurrence relation for N_k should refer to the previous values of itself and some values of $(p_k)_0^\infty$. Your recurrence for $(D_k)_0^\infty$ should refer to itself and $(N_k)_0^\infty$, the one for $(Q_k)_0^\infty$ should refer to itself and $(D_k)_0^\infty$, and the one for $(H_k)_0^\infty$ should refer to itself and $(Q_k)_0^\infty$.
- Using those recurrence relations, fill out this table, and solve the original question:

n	0	5	10	15	20	25	30	35	40	45	50
p_k											
N_k											
D_k											
Q_k											
H_k											

5 Challenge: Derangements

A *derangement* d of size n is a permutation of the numbers $1, 2, 3, \dots, n$ with no fixed points, so $d(k) \neq k$ for all numbers k. Let D_n be the number of derangements of size n. We're going to let $D(x)$ be the *exponential* generating function of the sequence $D_0, D_1, D_2, \dots$, which is the generating function for the sequence $\left(\frac{D_n}{n!}\right)_0^\infty$:

$$D(x) = \sum_{n=0}^{\infty} \frac{D_n}{n!} x^n = D_0 + D_1 x + \frac{D_2}{2!} x^2 + \frac{D_3}{3!} x^3 + \dots$$

As there are $n!$ permutations of size n in general, $\frac{D_n}{n!}$ is the fraction of the permutations of size n that are derangements.

This is called an exponential generating function because of the infinite series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

which converges for all $x \in \mathbb{R}$.

Problem 13 If $A(x)$ and $B(x)$ are exponential generating functions for the sequences $(a_n)_0^\infty, (b_n)_0^\infty$, and $C(x) = A(x)B(x)$ is the exponential generating function for $(c_n)_0^\infty$, find an expression for c_n in terms of $(a_n)_0^\infty$ and $(b_n)_0^\infty$.

Problem 14 Using the previous problem, find a rational function expression for $e^x D(x)$, solve for $D(x)$, and find a closed-form expression for its coefficients, $\frac{D_n}{n!}$. If you are familiar with infinite series convergence, use this expression to find $\lim_{n\to\infty} \frac{D_n}{n!}$.