Infinity III

Math Circle

February 9, 2020

1 More challenging problems

1. Show that if A countable, then $A \times A$ is countable. What about $A \times A \times A$?
Hint: Recall how we showed that $\mathbb{Q}$ is countable.

2. Show that a union of a countable number of countable sets is countable.
Hint: Is this problem similar to the previous one? Can you use a similar construction (yet again!)?

3. Show that if X is a countable set and $x_0 \in X$ then $|X| = |X - \{x_0\}|$. What if X is infinite, but not necessarily countable?
Hint: You can use the fact that every infinite set has a countable subset without proving it.

4. Let A, B be countable sets. X be a set of all bijections from A to B. Show that X is uncountable.

5. Let A be a set of all points in $[0, 1]$, and B be the set of all points in $(0, 1)$. Show that $|A| = |B|$.

2 Paradoxes

This part of the handout is about paradoxes related to infinity. First, let's do a couple of warm-up problems. What is wrong with the following conclusions?

- Here is a proof that $1 = 2$. Suppose that we have two numbers a and b which are equal, see that:

$$
\begin{aligned}
a &= b \\
a^2 &= ab \\
a^2 - b^2 &= ab - b^2 \\
(a-b)(a+b) &= b(a-b) \\
a + b &= b
\end{aligned}
$$

therefore if we select $a = b = 1$, we have that $2 = 1$.

- Let's do some math with i. Something is not right, is it?

$$\begin{aligned}
-1 &= i^2 \\
-1 &= \sqrt{-1}\sqrt{-1} \\
-1 &= \sqrt{(-1)(-1)} \\
-1 &= \sqrt{1} \\
-1 &= 1
\end{aligned}$$

Now we'll watch another video by Numberphile, which is called 'Infinity Paradoxes' and brings up 4 interesting paradoxes that occur when you think about the infinite.

`https://www.youtube.com/watch?v=dDl7g_2x74Q`

1. First, it is said that Hilbert's Hotel has an infinite number of rooms labeled $1, 2, 3, ...$ What is the cardinality of the set of all of the rooms in Hilbert's Hotel?

2. Now, initially the room starts off as full. What is meant mathematically by the hotel being full? Be specific.

3. The video shows how if you had 1 more guest show up, then they could be given a room by moving every single person down 1 room. By doing so, they show that there is a bijection between N and $\mathbb{N} \cup \{0\}$. Can you explain this connection between the hotel and the bijections?

4. Let's say that the patrons of Hilbert's Hotel are tired of moving, and they pressure the Governor to pass a new law. This new laws say that in order to make room for new guests in Hilbert's Hotel, you can't move more than %10 of the patrons. What is the maximum number of new guests that you can still accept?

   The third paradox is a paradox of probability. When you throw a dart at a dartboard there is a probability 0 that it will hit any particular point. Yet, it has to hit somewhere!

5. Suppose that the entire dart board has an idea of 1 meter squared, and you throw the dart so that it is just as likely to hit any one point as any other point. What are the chances that you'll hit a section of the board which is $p \leq 1$ meter squared?

6. What's the area of a single point? What are the chances that you'll hit a single point?

7. If you consider the sum of the chances of hitting every single point on the board you would get an infinite sum of the form $0 + 0 + 0 + \ldots$ This infinite sum is equal to zero. That means that your chances of hitting anywhere on the board must be 0. Is the previous argument a paradox, or a contradiction? Why?

   Let's recall the final paradox, where you play this double or nothing game.

8. Recall the expected value we learned last quarter. Let's say that I'll roll a fair die, and I'll give you 1 if it's a one, 2 if it's a two, etc. What is the expected value of your payoff from this game?

9. Suppose that I change the rules of the game so that if the roll is prime I give you 1, and if it's not, I'll take 1 from you. What is the expected payout of this game?

10. Prove that the expected value of the infinite double or nothing game is infinite.

11. Prove that if you have to pay an amount M to play the double or nothing game, the game is in your favor. No matter what M is (i.e, no matter what M is, your expected winnings in this game is > 0).