Tropical Orthogonal Representations

Los Angeles Math Circle, May 22, 2016
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1 Orthogonal Representations of Graphs

A **graph** G consists of a set V of **vertices** and a set E of **edges** between pairs of vertices (with at most one edge between each pair of vertices). Two vertices connected by an edge are said to be **adjacent**. Our goal is to find a way to represent graphs efficiently using vectors.

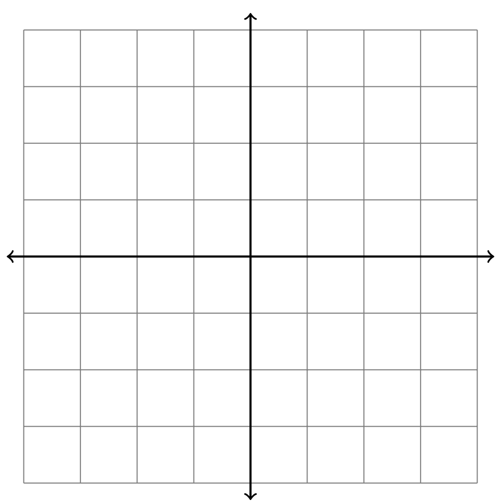
1.1 Vectors

A vector $\mathbf{v}$ is just a list of numbers (the **components** of the vector) written in a column inside of square brackets. For example,

$$\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 3.2 \\ -6 \\ 0 \\ -6 \end{bmatrix}.$$

We write $\mathbb{R}^n$ for the set of all vectors with n real components. A vector in $\mathbb{R}^n$ can be visualized as an arrow in n-dimensional space from the origin to the point with coordinates equal to the components of the vector.

1. Illustrate the vectors $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.



The dot product of two vectors in $\mathbb{R}^n$ is defined as follows:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Two vectors are **perpendicular** or **orthogonal** (meaning that they form a right angle) if and only if their dot product is 0.

2. Compute the dot product of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ to determine whether or not they are orthogonal.

3. Find a vector in $\mathbb{R}^3$ with no zero components which is orthogonal to $\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$.

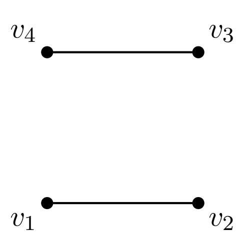
1.2 Orthogonal Representations of Graphs

An **orthogonal representation** of a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ is a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of vectors in some $\mathbb{R}^d$ such that

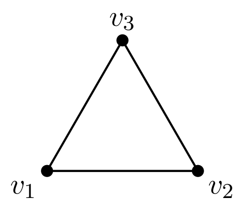
$$\mathbf{v}_i \text{ is orthogonal to } \mathbf{v}_j \text{ if and only if } v_i \text{ is adjacent to } v_j.$$

(This implies that none of the vectors $\mathbf{v}_i$ can be the zero vector, with all entries equal to 0.)

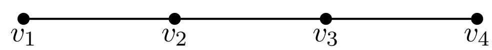
4. Find vectors in $\mathbb{R}^2$ which form an orthogonal representation of the graph below.



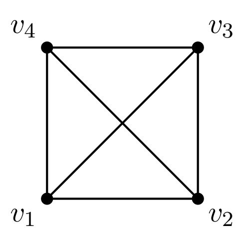
5. Find an orthogonal representation of the graph below in some $\mathbb{R}^d$, with d as small as possible.



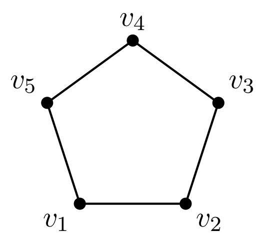
6. Find an orthogonal representation of the graph below in some $\mathbb{R}^d$, with d as small as possible.



7. Find an orthogonal representation of the graph below in some $\mathbb{R}^d$, with d as small as possible.



8. Find an orthogonal representation of the graph below in some $\mathbb{R}^d$, with d as small as possible.



2 Tropical Arithmetic

The real numbers together with infinity (∞) constitute the **tropical numbers**. In tropical arithmetic, we define addition and multiplication operations on the set of tropical numbers as follows. The **tropical sum** of two numbers is their minimum:

$$x \oplus y = \min(x, y)$$

while the **tropical product** of two numbers is their sum:

$$x \odot y = x + y.$$

2.1 Tropical Orthogonality

Recall that the "roots" of a tropical polynomial such as $(3 \odot x^2) \oplus (4 \odot x) \oplus (7)$ are the x-values for which the minimum indicated by the tropical addition is attained by at least two of the terms. (For this polynomial, equivalent to $\min(2x + 3, x + 4, 7)$, the roots are 1 and 3, the x-coordinates of the "corners" of the polynomial's piecewise-linear graph.)

We define orthogonality of tropical vectors in a similar way: two vectors $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ in $\mathbb{T}^n$ (the set of all vectors with n tropical components) are said to be **orthogonal** if the minimum indicated by the tropical dot product

$$(a_1 \odot b_1) \oplus (a_2 \odot b_2) \oplus \cdots \oplus (a_n \odot b_n) = \min(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

is attained by at least two of the terms.

9. Find three different tropical vectors which are orthogonal to the tropical vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

10. Find all tropical vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which are orthogonal to the tropical vector $\begin{bmatrix} 3 \\ 5 \\ \infty \end{bmatrix}$.

2.2 Tropical Orthogonal Representations of Graphs

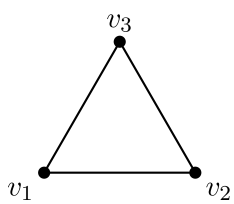
A **tropical orthogonal representation** of a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ is a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of tropical vectors in some $\mathbb{T}^d$ such that

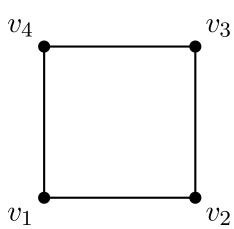
$$\mathbf{v}_i \text{ is orthogonal to } \mathbf{v}_j \text{ if and only if } v_i \text{ is adjacent to } v_j.$$

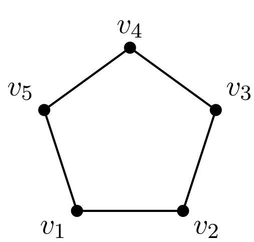
(This implies that none of the vectors $\mathbf{v}_i$ can be the "zero" vector, with all entries equal to ∞.)

Tropical orthogonal representations of graphs can be easier to find than standard orthogonal representations of graphs, once you get used to looking for them. However, tropical orthogonal representations usually require approximately the same number of vector components as standard orthogonal representations, which makes them interesting! The big question is, *how many components do (tropical) vectors need to have in order to represent the adjacency in a graph?*

11. Find a tropical orthogonal representation of the **cycle graph with** n **vertices** using vectors in some $\mathbb{T}^d$, with d as small as possible.

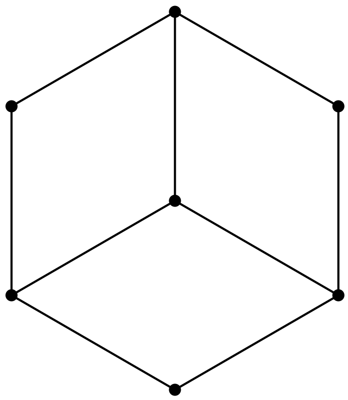






⋮

12. Can you find a tropical orthogonal representation of the graph below using vectors in $\mathbb{T}^4$? You may label the vertices in whichever order you like.



13. Can you find a method for obtaining a tropical orthogonal representation of any **tree graph** using vectors in $\mathbb{T}^3$? A "tree" is a graph that contains no cycles.