Lesson 5: Graphs and Geometry V

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Problem 1.
An Eulerian path in a graph is a path which passes through every edge exactly once. Show that there exists an Eulerian path in a given graph if and only if the graph is connected and all but exactly two vertices have even degrees.

Proof. Suppose there is an Eulerian path. Then the only two vertices which have odd degrees are the start and end of the path. If we already know that all but two vertices have even degrees, we can add a virtual edge between the two odd vertices, and then use the theorem about then existence of an Eulerian cycle. □

Problem 2.
Show that the segment connecting the midpoints of the opposing sides of a parallelogram goes through the intersection of its diagonals.

Proof. If $ABCD$ is a parallelogram, M is the midpoint of AB and N is the midpoint of CD, then $AMCD$ is also a parallelogram and thus MN goes through the midpoint of AC. □

Problem 3.
Show that the angle bisectors of a triangle intersect at the same point.

Proof. Take the triangle $\triangle ABC$. Let AD_A and BD_B be angle bisectors and let I be its intersection. Let H_A, H_B, H_C be the feet of the altitudes from I to BC, AC and AB respectively. Since $\angle IH_BA = \angle IH_CA = 90°$ and $\angle H_BAI = \angle H_CAI$ we get $\triangle IH_BA = \triangle IH_CA$ and thus $IH_B = IH_C$. Similarly looking at the angle bisector BD_B we get $IH_A = IH_C$. Therefore $IH_A = IH_B$. Then $\triangle IH_BC$ and $\triangle IH_BA$ are two right triangles with two equal sides, which means they are congruent. Therefore $\angle H_BCI = \angle H_ACI$ and so CI is the angle bisector of the $\triangle ABC$, and thus all angle bisectors intersect at point I. □