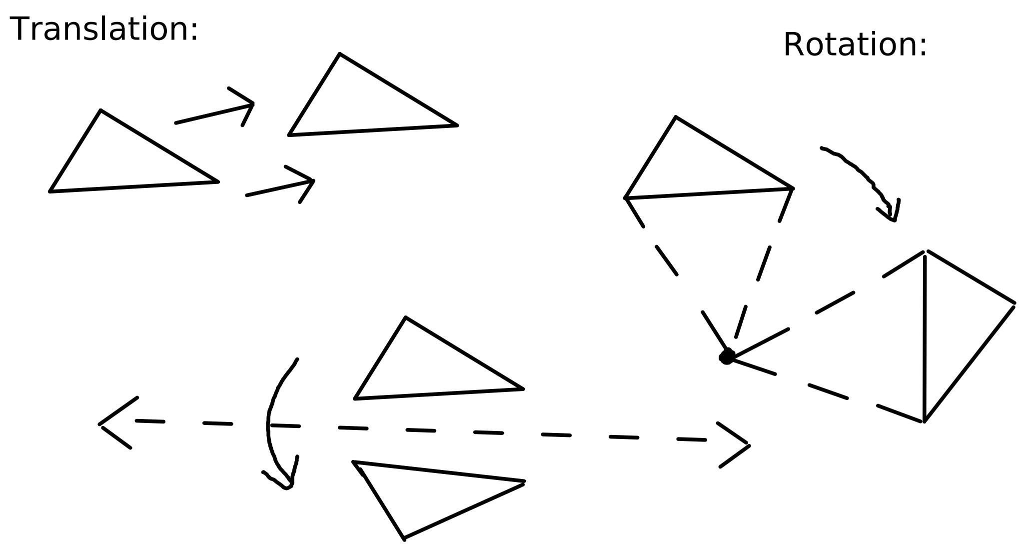
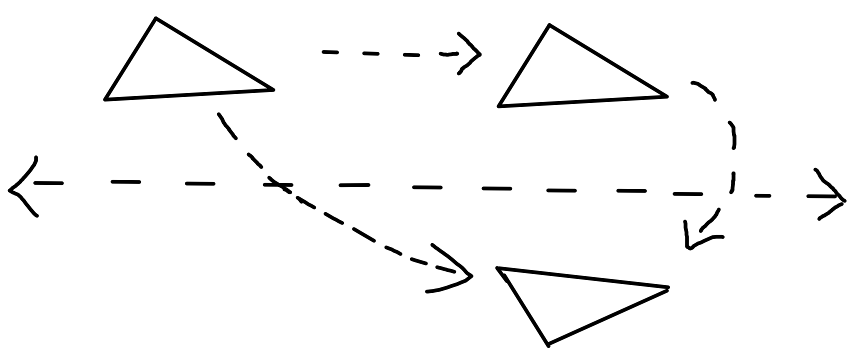
Motions of the Plane

Last week we discussed three types of motions of the plane – translations, rotations, and reflections.



To these we must add a fourth kind of motion – glide reflection:



(Exercise 8 on the last worksheet asked you to prove that a glide reflection is not the same as a translation, rotation, or reflection. If you haven't done that problem, you should do it now!)

Classifying Motions of the Plane

Theorem. *Every motion of the plane is a translation, rotation, reflection, or glide reflection.*

We will not write out a complete proof here, but the needed ingredients are the following facts:

1. Any movement of a triangle to a congruent triangle is achievable by composing the above types of motions.
2. A motion is completely determined by its action on any triangle (three noncollinear points).
3. Any composition of the above types of transformations is again one of the above types.

These three statements together imply that every motion of the plane is one of the four given types as follows:

- Given any motion of the plane, consider its action on a triangle (any triangle you want).
- Achieve this motion by composing motions of the four kinds above, which is possible by statement 1.
- The composition is the same as the given motion because they do the same thing to the triangle by statement 2.
- The composition is equal to one of the four given types by statement 3.

Another theorem of interest is the following:

Theorem 1. *Every rigid transformation of the plane is the composition of at most three reflections.*

Problem Solving with Motions of the Plane

Motions of the plane can be helpful in solving the following problems:

1. Given a line ℓ and points A, B not on the line, with line AB not parallel to ℓ, ...
   (a) Find the point M on ℓ such that the sum of the distances to A, B is minimal.
   (b) The difference between the distances to A, B is maximal.
2. Given a point O contained in the interior of triangle ABC, construct a line segments with endpoints on the perimeter such that O is the midpoint of the line segment.
3. Given a point in the plane and two parallel lines, none of which intersect, find an equaliteral triangle with one vertex at the point, and one vertex on each of the two lines.
4. In trapezoid $ABCD$, where $AD \parallel BC$, points M, N which are the midpoints of AB and CD. If the line MN intersects the lines AB and CD in equal angles, prove that the trapezoid is isosceles (i.e. the non-parallel sides have the same length).
5. Given five straight lines, show how to inscribe a pentagon in a circle such that the sides of the pentagon are perpendicular to the given lines.