PROBLEM SESSION 3

LAMC OLYMPIAD GROUP, WEEK 5

Geometrical constructions. Here are a few constructions that you should always consider adding to your figure:

(1) If two circles intersect at points P and Q, **draw the segment** PQ.
(2) If a circle centered at O tangent to a line ℓ at a point A, **draw the radius** OA (which is perpendicular to ℓ).
(3) If two circles are tangent at a point A, **draw the common tangent to the two circles** through A (and also draw the radii at A).
(4) Given a right triangle $\triangle ABC$ with $\angle A = 90°$, if the middle M of the segment BC is already on the figure, **draw the median** AM (note that $AM = BM = CM$).
(5) If two line segments are involved in a "nice" angle configuration, **extend their lines until they intersect** (especially if you can form an isosceles or right triangle).
(6) In an isosceles triangle, **draw the median = height = bisector** from the top vertex!
(7) In a parallelogram, **draw the diagonals** (recall that they intersect at their middle points).
(8) If your figure already involves two parallel lines, try **drawing another parallel line** to them through a convenient point.
(9) In a problem about areas, **draw heights** wherever appropriate!

Among others, these constructions will allow you to do **angle chasing** on your figure (by which you try to compute all angles using isosceles triangles, parallel lines, circles, etc.).

Problem 1. Let $ABCD$ be a parallelogram of area 1, and M a point inside the segment AD. Compute the area of $\triangle BMC$.

Problem 2. **(a)** Let $\triangle ABC$ be an acute triangle, and let D, E, F be the projections of A, B, C on sides BC, CA, AB respectively. Also construct DE, EF, FD on the picture. Compute all angles that show up, in terms of $\hat{A}, \hat{B}, \hat{C}$.

(b) Let $\triangle ABC$ be a triangle, and let $\mathscr{I}$ be the inscribed circle circle. If D, E, F are the contact points of $\mathscr{I}$ with the sides BC, CA, AB respectively, compute all angles of triangle $\triangle DEF$.

Problem 3. Consider a spreadsheet with n rows and n columns, where n is an *even* number. Write the numbers $1, 2, \ldots, n^2 - 1, n^2$ in the n^2 cells, from left to right, and then from top to bottom, as in

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Now color each cell of the spreadsheet with red or blue, such that each row and each column has exactly $n/2$ red and $n/2$ blue cells. Show that the sum of the entries of the red cells equals the sum of the entries in the blue cells.

Problem 4. On a board there are n positive integers written. A move consists of choosing three numbers a, b, c from the board, all of the same parity but not all equal, and then replacing them with $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$. Show that by repeatedly applying such moves, at some stage the process must end.