

Infinity II

Advanced 1

February 2, 2020

Summarizing what we already know about sizes of sets, we can write the following definitions.

Definition 1 A set A is called **finite** with cardinality n if there is a bijection between A and \mathbb{N}_n , for some n .

Definition 2 A set A is called **infinite** if it is not finite.

Last time, we learned that there are quite a few sets that can be put into a bijection with the set of natural numbers. We call such sets countable sets.

Definition 3 A set A is called **countable** if there is a bijection between A and \mathbb{N} .

Also recall from last class the following definitions. Let A and B be two sets.

Definition 4 A function $f : A \rightarrow B$ is called a **injective (1-to-1)**, if for any $a, b \in A$:

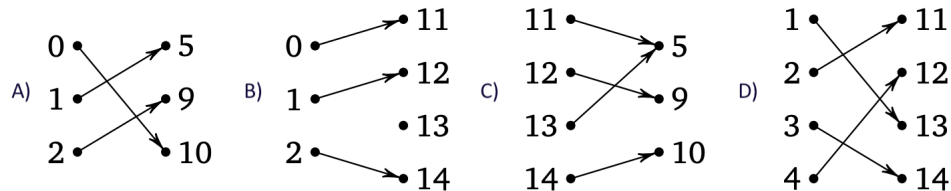
$$a \neq b \implies f(a) \neq f(b)$$

Definition 5 A function $f : A \rightarrow B$ is called a **surjective (onto)**, if for any $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Definition 6 A function $f : A \rightarrow B$ is called a **bijection**, if it is both surjective and injective (1-to-1 and onto).

1. Now let's do some warm-up problems.

(a) For each of the following functions, decide if it is injective, surjective, or a bijection.



(b) Can you find a bijection between the rational numbers (the fractions) and the natural numbers? In other words, prove that \mathbb{Q} is countable.

(c) Can you find a graphical bijection between the points on the line segment $[0, 1]$ and $[0, 2]$? I know that you can find a formula to do it, but I want you to use a geometric argument.

(d) Using a picture, show that there is a bijection between the points in $[0, 1]$ and between any finite segment of the real line, that is $[a, b]$ for any $a, b \in \mathbb{R}, a < b$.

(e) Using a picture, can you find a bijection between all of the points on the real line and only the points in $(0,1)$?

(f) Do you think that there is a bijection between \mathbb{R} and $\mathbb{R}^2 = (\mathbb{R}, \mathbb{R})$? If so try and come up with one. If not, explain (or better yet prove!) why not.

2. At this point you have probably seen time and time again that you can find bijections between lots and lots of different sets. The obvious question to ask here is, given two infinite sets, can you always find a bijection between them? Let's watch a video from Numberphile, called "Infinity is bigger than you think" with a surprising answer.

<https://www.youtube.com/watch?v=e1v0Zm0d4H0>

Let's digest that proof a little bit more. You can't try all possible bijections, so your only hope is to come up with an algorithm to 'automatically' rule out possible candidate bijections.

The trick is this, suppose that we had a candidate for a bijection, then we could write it like this:

\mathbb{N}	$[0, 1)$
1	$\rightarrow .032591\dots$
2	$\rightarrow .510539\dots$
3	$\rightarrow .000034\dots$
4	$\rightarrow .328641\dots$
5	$\rightarrow .889813\dots$
6	$\rightarrow .141599\dots$
\vdots	\vdots

Now look at the $[0, 1)$ column and focus on only the number along the diagonal,

\mathbb{N}	$[0, 1)$
1	$\rightarrow .\underline{0}32591\dots$
2	$\rightarrow .5\underline{1}0539\dots$
3	$\rightarrow .00\underline{0}034\dots$
4	$\rightarrow .328\underline{6}41\dots$
5	$\rightarrow .8898\underline{1}3\dots$
6	$\rightarrow .14159\underline{9}\dots$
\vdots	\vdots

Let m be the number in $[0, 1)$ which is formed by taking every digit on the diagonal, so for this candidate bijection, $m = .010619\dots$ Now, let \tilde{m} be the number that you get if you add 1 to each digit of m modulo 10, so $\tilde{m} = .121720\dots$ What can you say about \tilde{m} ? Can it appear in the right hand side of the list? What does this tell you about this candidate bijection?

- (a) Use this to prove that there is no bijection between \mathbb{N} and $[0, 1)$.

Now we are ready to introduce another definition:

Definition 7 A set A is called **uncountable** if it is neither countable nor finite.

- (b) Prove that there is no bijection between \mathbb{N} and \mathbb{R} . Since \mathbb{R} is not finite, it is thus **uncountable**.

Definition 8 For two sets A and B , we write $|A| < |B|$ if there exists an injective function from A to B , but no bijective function.

- (c) Prove that this definition corresponds to what you know about comparing sizes (cardinalities) for finite sets.

- (d) Prove that $|\mathbb{N}| < |\mathbb{R}|$.

Definition 9 Let X be a set. The set $P(X)$, called the **power set** of X is the set of all subsets of X .

- (e) Write down the power sets of $\{1, 2\}$ and $\{33, 44, 55\}$.
- (f) Suppose $|X| = 4$. How many elements are there in $P(X)$? What if $|X| = 5$? How about $|X| = n$?
- (g) Prove if you have a nonempty set \mathcal{X} , then $|\mathcal{X}| < |P(\mathcal{X})|$.
Hint: After you figure how to solve the problem for finite sets, proceed similarly to what you did in 2(a). Suppose you have a ‘bijection’ $f : X \rightarrow P(X)$, can you construct a subset of X that is not hit by f ?

- (h) Do you think that there is a set \mathcal{Y} such that $|\mathbb{N}| < |\mathcal{Y}| < |\mathbb{R}|$? Explain.

Challenge Questions

- (a) Can you take an interval of length 1, cut it up into tiny pieces, and move those pieces around so that they covers every rational number? What if your initial interval is of length $\frac{1}{2}$? $\frac{1}{4}$? What does this tell you? Can you do the same thing for \mathbb{R} ?
- (b) Can you find a bijection between \mathbb{R} and \mathbb{R}^n where n can be any natural number?
- (c) Can you prove that $|\mathbb{R}| < |\mathcal{F}|$ where \mathcal{F} is the set of all functions from \mathbb{R} to \mathbb{R} ?