Lesson 4: Graphs and Geometry IV

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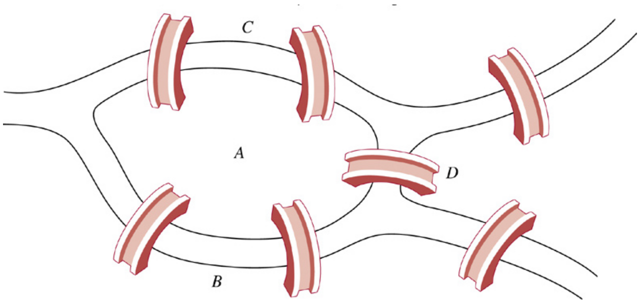
Problem 1.
a) The city of Köningsberg consists of 4 islands and 7 bridges, connecting them as shown on the picture. Is it possible to walk around the city starting and ending at the same island and using every bridge exactly once?



b, c) An Eulerian cycle in a graph is a cycle which passes through every edge exactly once. Show that there exists an Eulerian cycle in a given graph if and only if the graph is connected and all vertices have even degrees. Here the "only if" direction is part b) and the "if" direction is part c).

Proof. This is a very standard well-known theorem. The proof is rather lengthy to write down, but can be easily found in existing references. □

Problem 2.
Two squares of the chessboard are called connected if and only if it is possible two make a knight's jump from one to the other. Is it possible to walk around the chessboard with a knight while going along every possible connection exactly once? It is not required to start and end at the same spot.

Proof. No, this is not possible. Suppose it was. Then for every square but the starting or ending, every time we jumped into it we had to jump out. So at most 2 squares can have an odd number of possible jumps. But there are at least 8 such squares on the chessboard – the neighbors of the corner cells. □

Problem 3.
In a certain country every city has 3 roads connected to it. Show that it is possible to orient every road in such a way that no city has all three roads coming in or out of it.

Proof. Since the total sum of degrees is even, the number of vertices in this graph has to be even. Let us arbitrarily split the vertices into pairs, and add a new "virtual" edge between each pair. The resulting graph might be technically a multigraph since some of the newly connected pairs of vertices might have been already connected, but that ultimately does not matter. In a new graph every vertex has degree 4. Then by problem is this graph has an Eulerian cycle. Now let us put an arrow on every edge while we trace out the cycle. Then each vertex has two arrows in and two out. Then once we remove the "virtual" edges we added each vertex has either two in one out of two out one in, and we are done. □

Problem 4.
A quadrilateral $ABCD$ is called a parallelogram if $AB \parallel CD$ and $AC \parallel BD$. We will show some other defining properties of parallelograms:

a) Show that $ABCD$ is a parallelogram if and only if $AB = CD$ and $AC = BD$.

b) Show that $ABCD$ is a parallelogram if and only if $AB = CD$ and $AB \parallel CD$.

c) Let O be the intersection of the diagonals of $ABCD$. Show that $ABCD$ is a parallelogram if and only if $AO = OC$ and $BO = OD$.

d) Show that the diagonals of a parallelogram $ABCD$ are perpendicular if and only if all of its sides are equal. Such a parallelogram is called a rhombus.

Proof. We will optimize the solution by solving all the parts at the same time. First of all, let me show a parallelogram has all the properties outlined. Indeed, suppose $AB \parallel CD$ and $AD \parallel CB$. Then $\angle ABD = \angle BDC$, $\angle BAC = \angle DCA$, $\angle DAC = \angle ACB$ and $\angle ADB = \angle DBC$. Then the SAS test tells us that $\triangle ADC = \triangle CBA$, which implies $AB = CD$ and $AD = BC$. This is parts a) and b). Let O be the intersection of diagonals. Then by the SAS test we know have $\triangle AOB = \triangle COD$, which implies part c). Now let us do the other directions.

a) Suppose $AB = CD$ and $AD = BC$. Then by SSS $\triangle ADC = \triangle CBA$, which implies $\angle BAC = \angle DCA$ and $\angle DAC = \angle ACB$ which in turn imply $AB \parallel CD$ and $AD \parallel CB$.

b) $AB \parallel CD$ implies that $\angle DCA = \angle CAB$ and $\angle BDC = \angle DBA$. Then together with $AB = CD$ we get $\triangle AOB = \triangle COD$, which implies $BO = DO$ and $AO = OC$. Then we can conclude that $ABCD$ is a parallelogram by part c), which is proved below.

c) Suppose $BO = DO$ and $AO = OC$. Then by SAS we have $\triangle AOB = \triangle COD$ and $\triangle AOD = \triangle COB$, which imply $AB = CD$ and $AD = BC$, and we are done by part a)

d) Suppose $AC \perp BD$. Then in $\triangle ABD$ we get that the median AO coincides with the altitude. Then $AD = AB$, implying that all the sides are equal. Conversely, if all the sides are equal, then the median must coincide with the altitude, so the diagonals are perpendicular. □

Problem 5.
Let $\triangle ABC$ be isosceles with $AB = AC$ and $\angle BAC = 30°$. Let AD be the median, P be a point on AD and Q be a point on AB such that $PQ = PB$. Find $\angle PQC$.

Proof. Note that since PD is both the altitude and the median in $\triangle BPC$ we have $PC = PQ = PB$. Let

$$\angle PCQ = \angle PQC = \alpha$$

$$\angle PQB = \angle PBQ = \beta$$

$$\angle PBC = \angle PCB = \gamma$$

Then $\alpha + \beta + \gamma = 90^\circ$ and

$$\beta + \gamma = \angle ABC = \frac{180^\circ - 30^\circ}{2} = 75^\circ$$

Then $\angle PQC = 15^\circ$. $\square$