Infinity I

Advanced 1

January 26, 2020

1. Today we are going to talk about infinite sets.

   When we talk about infinite sets, we have to use a special language. The reason for this is because when you talk about infinities using the normal algebra that you know and love, you run into some 'problems.'

   Suppose that infinity was a number, a number which satisfied the identity: $\infty = \infty + 1$. Using the normal rules of arithmetic, show the following:

   (a) Not only does $\infty = \infty + 1$, but $\infty = \infty + C$ for any integer C.

   (b) Show that $\frac{\infty}{\infty} = 1$

(c) Show also that $\frac{\infty}{\infty} = 2$

(d) Show that $\infty = 1$

(e) Show that $0 = 1$

I'm sure that people have told you that "∞ is not a number," but you may have wondered why. Hopefully these past exercises help explain why that is the case.

With this in mind, mathematicians developed a different way of comparing the number of things (also known as the cardinality) in two groups. Suppose that you have two different sets of objects and you want to figure out if both groups have the same cardinality. What can you do? The most natural thing to do would be to count them, but what if you didn't know how to count? You could try the following:

Suppose that you had a bunch of ugly Christmas sweaters and a bunch of guests over for a holiday party, and you wanted to make sure that you had enough ugly sweaters for all of your guests. You could one by one hand out the sweaters to people who are not already wearing one and continue until you run out of sweaters or dinner guests. If every person has a sweater, and every sweater a person, then you have exactly the same number of sweaters as people. No counting required.

This idea of pairing things up turns out to be useful and thus has a name. A **bijection** between two sets is a rule (or function) that assigns elements in one set to elements in another set, such that one element in each set is paired with exactly one element in the other set and vise versa. Bijections are useful because they are a way to avoid counting and even work for infinite sets!

2. Let's warm up by finding some bijections between finite sets.

   (a) Can you find a bijection between the even numbers between 1 and 100 and the odd numbers between 1 and 100?

   (b) If you are in a crowded classroom where every seat is occupied and each seat can only fit one person, can you find a bijection between the sitting people and seats?

(c) We'll give a proof to the fact that $\binom{n}{2}$ equals the sum of the first $n-1$ integers. Can you find a bijection between 2-element subsets and components of the sum to show this? Use this triangle below as a hint!

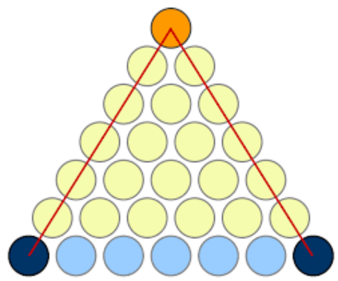

Figure 1: Taken from https://mathoverflow.net

(d) What about the whole numbers from 1 to 10 and the perfect squares between 9 and 144?

(e) Can you find a bijection between a set with n elements, and the set $\mathbb{N}_n = \{1, 2, 3, \ldots, n-1, n\}$? This is actually how we define the cardinality of a finite set.

Definition 1 *A set A has cardinality n if it can be put into bijection with $\mathbb{N}_n$. This is usually written as $|A| = n$*

(f) Prove that if you have two finite sets of the same cardinality, then they can be put into bijection with each other. Remember, there might be many possible bijections, we just need to find one.

(g) Conversely, if two (finite) sets can be put into bijection with each other, they have the same cardinalities. This is what you used in (c).

3. Okie dokie, now that we have found some bijections between finite sets, let's find some bijections between infinite sets!

   (a) Can you find a bijection between all of the even numbers $E = \{0, 2, 4, 6, \dots\}$ and all of the odd numbers $O = \{1, 3, 5, 7, \dots\}$?

   (b) Can you find a bijection between all of the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ and all of the natural numbers and zero $\{0, 1, 2, 3, \dots\}$?

   (c) What about $\mathbb{N}$ and $\mathbb{N} + 42 = \{43, 44, 45, \dots\}$?

(d) What about $\mathbb{N}$ and the perfect squares $\{1, 4, 9, 16, \dots\}$?

(e) What about between $\mathbb{N}$ and $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ If you can, then that would be pretty incredible. That would mean that there are exactly as many positive whole numbers as there are integers. Surprising?

(f) Can you find a bijection between $\mathbb{N}$ and all pairs of integers $\mathbb{Z} \times \mathbb{Z}$?

(g) There is an enemy among us. As we speak, there is a submarine somewhere on the number line. We don't know where it started, only that it started at some integer, and that is is moving with a constant integer velocity. For example, it could have started at $+4$ and could have a velocity of 1, or it could have started at -2502351 and it could be moving with a velocity of -2359. Every second we can release a depth charge at an integer, and it will immediately explode and destroy the sub if it is there. Can you come up with a plan to eventually destroy the submarine?

(h) Can you find a bijection between the rational numbers (the fractions) and the natural numbers? How about between the natural numbers and the real numbers?

(i) Can you find a bijection between the natural numbers, and all finite sequences of natural numbers?