Lesson 1: Graphs and Geometry

Konstantin Miagkov

June 26, 2019

Remark 1.
In the beginning of this class one should define a graph and show how it generalizes various natural situations – cities and roads, airports and plane routes, people and friendships, etc. It is also necessary to remind everyone of the triangle equality test – that (and some axioms which we are sweeping under the rug) is the only input we will be assuming to build up Euclidean geometry. In the problems 5 and 6 from this handout, however, the students should be allowed to use any geometry knowledge they have – it is designed to gauge the level of their prior familiarity with geometry.

Problem 1.
There are 100 cities in a country, and each road connects two of those cities. If each city has 4 roads going out of it, how many roads are there in total?

Proof. Since every road ends in exactly two cities, the total number of roads is equal to $100 \cdot 4/2 = 200$. □

Problem 2.
There are 15 cities in a country, each connected to at least 7 other cities by roads. Show that it is possible to drive from any city to any other city using the roads.

Proof. Pick two arbitrary cities. If the two cities are connected via a direct road, then we are done. If they are not, each is directly connected to 7 cities out of remaining 13. Then by pigeonhole principle there is a city they are both directly connected to, and thus there is a path between them. □

Problem 3.
Prove that in any group of 6 people there are either 3 people who all know each other, or 3 people none of whom know each other. All acquaintances are assumed to be mutual. (Meaning that if John knows Pete, than Pete necessarily knows John)

Proof. Let's take an arbitrary person A. Since there are 5 people remaining, A either knows three of them or does not know three of them. Without loss of generality assume A known three people B, C, D. If any two of them know each other, say B and C, then A, B, C is the triplet where everyone knows each other. If none of B, C, D know each other, then that triple is what we are looking for, and we are done. □

Definition 1.
For the next problem, let us introduce everyone (or remind of) a few concepts. 1. In a triangle ABC a *median* from a vertex to the opposing side is a segment connecting the vertex to the middle of the opposing side.

2. In the same setting, an *angle bisector* of an angle of a triangle is the segment from the vertex to the opposing side which divides the angle at the vertex into two equal parts.
3. Finally, an *altitude* from a vertex is a segment to the line containing the opposing side which makes a right angle with that line.

Problem 4.
a) Show that if $\triangle ABC$ is isosceles with $AB = BC$, then the median, and altitude from vertex B to AC coincide. For this problem, you are only allowed to assume triangle equality tests.

Proof. Let BM be the median. Then $\triangle ABM = \triangle CBM$ by the SSS test, which implies $\angle AMB = \angle CMB$. Since those angles add up to $180°$, we get $\angle AMB = \angle CMB = 90°$ and AM is also the altitude. □

b) In the same setting, show that the angle bisector and the altitude coincide. Conclude that all 3 of the altitude, angle bisector and the median coincide.

Proof. As in part a), $\triangle ABM = \triangle CBM$. Then $\angle ABM = \angle CBM$, and the median is also the angle bisector, so all three coincide. □

Problem 5.
$\triangle ABC$ is isosceles with $AB = BC$. It is known that one of the sides AB and AC is 7, and the other is 3. Which is which?

Proof. Suppose $AB = BC = 3$. Then $AB + BC = 6 < 7 = AC$, which contradicts the triangle inequality. □

Problem 6.
Let A, B, C be points on a circle ω. Let P be a point such that the line PB is tangent to ω. Also let A_1 be the foot of the altitude from P to AB, and C_1 be the foot of the altitude from P to CB. Show that $A_1C_1 \perp AC$.

Proof. Let T be the intersection of the lines A_1C_1 and AC. Since $PC_1 \perp BC$ we have

$$\angle TC_1B = 90° - \angle PC_1A_1$$

Since $\angle PA_1B = \angle PC_1B = 90°$ we know that the quadrilateral PC_1BA_1 is cyclic, and thus $\angle PC_1A_1 = \angle PBA_1$. Since PB is tangent to ω, we have $\angle PBA_1 = \angle ACB$. Therefore

$$\angle TC_1B = 90° - \angle ACB$$

which means $\angle C_1TC = 90°$. □