Lesson 9: Graphs and Geometry

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Problem 1.
There are 100 cities in a country, and each road connects two of those cities. If each city has 4 roads going out of it, how many roads are there in total?

Problem 2.
There are 15 cities in a country, each connected to at least 7 other cities by roads. Show that it is possible to drive from any city to any other city using the roads.

Problem 3.
Prove that in any group of 6 people there are either 3 people who all know each other, or 3 people none of whom know each other. All acquaintances are assumed to be mutual. (Meaning that if John knows Pete, than Pete necessarily knows John)

Definition 1.
For the next problem, let us introduce everyone (or remind of) a few concepts. 1. In a triangle ABC a *median* from a vertex to the opposing side is a segment connecting the vertex to the middle of the opposing side.
2. In the same setting, an *angle bisector* of an angle of a triangle is the segment from the vertex to the opposing side which divides the angle at the vertex into two equal parts.
3. Finally, an *altitude* from a vertex is a segment to the line containing the opposing side which makes a right angle with that line.

Problem 4.
a) Show that if $\triangle ABC$ is isosceles with $AB = BC$, then the median, and altitude from vertex B to AC coincide. For this problem, you are only allowed to assume triangle equality tests.

b) In the same setting, show that the angle bisector and the altitude coincide. Conclude that all 3 of the altitude, angle bisector and the median coincide.

Problem 5.
$\triangle ABC$ is isosceles with $AB = BC$. It is known that one of the sides AB and AC is 7, and the other is 3. Which is which?

Problem 6.
Let A, B, C be points on a circle ω. Let P be a point such that the line PB is tangent to ω. Also let A_1 be the foot of the altitude from P to AB, and C_1 be the foot of the altitude from P to CB. Show that $A_1C_1 \perp AC$.