Lesson 3: Graphs and Geometry III

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May 17, 2019

Problem 1.

a) Let ABC be a triangle. Show that if the median and the altitude from A coincide, then ABC is isosceles.

Proof. Suppose AM is the median and also the altitude from point A. Then $BM = CM$ and $\angle AMB = \angle AMC = 90°$. Then $\triangle AMB = \triangle AMC$ by the SAS test, and therefore $AB = AC$ and $\triangle ABC$ is isosceles. □

b) Show the same if the angle bisector and the altitude coincide.

Proof. Same proof, except using the ASA test instead. □

c) Show the same if the angle bisector and the median coincide.

Proof. Let AM be the median and also the angle bisector. Let D be the point on the ray AM such that $AM = MD$. Then $\triangle AMC = \triangle DMB$ by the SAS test, and thus $\angle ADB = \angle MAC = \angle MAB$ and $AC = DB$. Since $\angle ADB = \angle MAB$ we have $AB = DB$, which implies $AC = AB$ and $\triangle ABC$ is isosceles. □

Problem 2.

In a quadrilateral $ABCD$ we have $AB = AD$ and $CB = CD$. Show that the diagonals of $ABCD$ are perpendicular.

Proof. The triangle ABC and the triangle ADC are congruent to each other since $AD = AB, DC = BC, AC = AC$. So $\angle DAC = \angle BAC, angleDCA = \angle BCA$. So CA bisects $\angle DAB$. $\triangle DAB$ is an isosceles triangle. We proved before that the angle bisector coincides with the altitude in an isosceles triangle. So $AC \perp DB$ □

Problem 3.

a) Let AM be the median of $\triangle ABC$. Show that if $AM = BM = CM$, then $\angle BAC = 90°$.

Proof. $\triangle BMA$ is isosceles, so $\angle ABC = \angle MAB$ and $\angle ACB = \angle MAC$. Adding those two we get $\angle ABC + \angle ACB = \angle BAC$, and since all the angles add up to $180°$ we get that $\angle BAC = 90°$. □

b) Show the converse: if $\angle BAC = 90°$, then $AM = BM = CM$ where AM is the median.

Proof. Suppose $AM > BM$. Then $\angle ABC > \angle MAB$ and $\angle ACB > \angle MAC$, which implies $\angle ABC + \angle ACB > \angle BAC$ and thus $\angle BAC$ cannot be $90°$. The case of $AM < BM$ is similar. □

Problem 4.
Can 9 line segments be drawn on a plane in such a way that each intersects exactly 3 others?

Proof. Consider the graph where the segments are vertices and two vertices are connected if the corresponding segments intersect. Then the sum of degrees of this graph would be 27. This is supposed to be twice the number of edges, but 27 is odd, contradiction. □

Problem 5.
In a certain country there are 2018 roads going out of every city, in such a way that all cities are connected by the roads network. Show that if any one road is closed for maintenance, all the cities are still connected.

Proof. Pick two scientist who are not friends with each other: a and b. There are $50-2 = 48$ more scientists other than them. a is friends with 25 people in the other 48 people and so does b. But $25+25 = 50$ which is two more than 48. By the Pigeonhole Principle, a and b must have two common friends c and d. Now we can sit a, b, c, d clockwise (or counterclockwise) on the round table and every two neighbors will know each other. □

Problem 6.
There are 50 scientists at a conference, and every scientist knows 25 others. Show that it is possible to find 4 scientists and sit them at a round table so that every two neighbors know each other.

Proof. Take any pair of scientists A, B who do not know each other. Then there are 48 scientists other than A and B, and each of A, B knows 25 of them. Then by pigeonhole there are two scientists C, D such that both A and B know them. Then A, C, B, D is a valid four-people configuration. □