Lesson 2: More graphs and geometry

Konstantin Miagkov

June 26, 2019

Remark 1.
Before this lesson, it is necessary to introduce parallel lines along with Euclid's axiom – through every point not on a given line there exists exactly one line parallel to the given one. From this axiom it is possible to deduce that two lines are parallel if and only if the alternate interior angles are equal for any transversal. Then we can use this fact to show that the sum of angles of any $\triangle ABC$ is $180°$ by drawing a line through A parallel to BC.

Problem 1.
Show that in a $\triangle ABC$ the angle supplementary to $\angle ABC$ equals the sum of angles $\angle BCA$ and $\angle BAC$.

Proof. This follows from the fact that sum of angles of a triangle is $180°$. □

Problem 2.
a) Suppose that in a $\triangle ABC$ we have $AB > BC$. Show that $\angle ACB > \angle BAC$. (Hint: Pick a point D on AB such $BD = BC$, and try to remember something about an isosceles triangle.)

Proof. Since $BD = BC$ we have $\angle BDC = \angle BDC$. By problem 1 we have

$$\angle BAC = \angle BDC - \angle ACD = \angle BCD - \angle ACD < \angle BCD < \angle BCA$$

□

b) Suppose that in a $\triangle ABC$ we have $\angle ACB > \angle BAC$. Show that $AB > BC$.

Proof. If $AB \leq BC$, then by part a) we have $\angle ACB \leq \angle BAC$, contradiction. □

Problem 3.
In $\triangle ABC$ it is known that $AB = BC$ and $\angle ABC = 108°$. Let D be the foot of the angle bisector of $\angle BAC$. Let E be the intersection of AC and the line through D perpendicular to AD. Show that $BD = BE$.

Proof. Let T be the intersection of lines AB and DE. Then $\angle TBD = 72°$. Since $\angle ABC = 108°$ and $AB = BC$ we have $\angle BAC = 36°$ and consequently $\angle BAD = 18°$. Then $\angle BDA = 36°$ and therefore $\angle BDT = 36°$. This implies that $\angle BTD = 72°$. Then $\angle TBD = \angle BTD = 72°$ and thus $BD = BT$. On the other hand, AD is both the altitude and the angle bisector in the triangle $\triangle ATE$, which means it $\triangle ATE$ is isosceles and $TD = DE$. Therefore $BD = DE$, and we are done. □

Problem 4.
Show that the number of states in the US with an odd number of neighboring states is even.

Proof. Consider the graph where each state of the US is a vertex, and two vertices are connected if and only if the corresponding states are neighboring. Let e be the number of edges in the corresponding graph. Then if we add up the the number of neighboring states for every state, we get $2e$, which is an even number. But in the sum of an odd number of odd numbers is odd, so the number of states with an odd number of neighbors must be even. □

Problem 5.
In a group of 10 people there are 14 pairs who hate each other. Show that it is still possible to assemble a friendly trio of people.

Proof. Let us count how many trio of people can be assembled in total, regardless of relationships. Some of you might remember that this number can be computed as 10 *choose* 3

$$\binom{10}{3} = \frac{10!}{7!\,3!} = 120$$

from one of last year's topics. If not, it is easy to get this number by hand: the first person in a triple can be chosen in 10 ways, the second in 9 and the third in 8. Except in a trio we do not care about the ordering of people, so in $10 \cdot 9 \cdot 8$ we overcount every triple 6 times – the number of ways to assign who was chosen first, who second and who third. So the total number of trios is $10 \cdot 9 \cdot 8/6 = 120$. Now let us count how many triples have at least one bad relationship in them. Every bad relationship spoils at most 8 triples, as there are 8 people one can add to make a triple with 2 people who don't like each other. With 14 bad relationships, that means at most $8 \cdot 14 = 112$ spoiled triples. But there are 120 in total, so at least 8 are good – and we needed to find at least one. □