Complex Numbers

Let's make a deal. Even though we all learned that when you square a number you don't get a negative number, let's pretend there is an "imaginary" number i such that $i^2 = -1$.

If i is to be a number, then we have to be able to add it to and multiply it by other numbers. Therefore, for example $2i$ is a number, $1 + i$ is a number, and so on.

A **complex number** is a number of the form $a+bi$ (or $a+ib$, which represents the same thing), where a, b are real numbers.

Examples:

- $1 + i$, $3 + 2i$, $5 - 6i$
- All real numbers are complex numbers. For example, $2 = 2 + 0i$.
- Multiples of i are called *imaginary numbers*, and they are complex numbers. For example $2i = 0 + 2i$.

We often use the letter z to stand for a complex number.

If $z = x + iy$, then we say that x is the **real part** of z, written $x = \operatorname{Re} z$, and y is the **imaginary part** of z, written $y = \operatorname{Im} z$:

$$z = x + iy \qquad \rightarrow \qquad \begin{array}{ll} x = \operatorname{Re} z, & \text{real part} \\ y = \operatorname{Im} z & \text{imaginary part} \end{array}$$

Complex Arithmetic

The arithmetic of complex numbers obeys (most of) the same rules as the arithmetic of real numbers. We usually treat i like a variable, with the exception that $i^2 = -1$.

Addition and Multiplication

To add two complex numbers, we add their respective real and imaginary parts:

$$(1+i) + (3+2i) = (1+3) + (i+2i) = 4+3i.$$

To multiply two complex numbers, we "multiply out" the expression:

$$\begin{aligned}
(1+i)(3+2i) &= 1(3+2i) + i(3+2i) \\
&= (3+2i) + (3i+2i^2) \\
&= (3+2i) + (3i-2) && \text{(since } i^2 = -1) \\
&= (3-2) + (2i+3i) \\
&= 1+5i. \\
(3+i)(3-i) &= 3(3-i) + i(3-i) \\
&= (9-3i) + (3i-(-1)) \\
&= (9+1) + (-3i+3i) \\
&= 10.
\end{aligned}$$

Complex Conjugation

An operation that the real numbers do not have is complex conjugation. When $z = a+bi$, the **complex conjugate** of z is defined to be

$$\bar{z} = a - bi.$$

The **modulus** (or absolute value) of a complex number $z = x+iy$ is defined to be

$$|z| = \sqrt{x^2+y^2}.$$

Exercise: Prove that $|z|^2 = z\bar{z}$.

Division

How can we divide by a complex number? When $a+bi$ is in the denominator, we can multiply top and bottom by the complex conjugate $a-bi$. For example:

$$\frac{2}{1+i} = \frac{2(1-i)}{(1+i)(1-i)} = \frac{2-2i}{1-i^2} = \frac{2-2i}{2} = 1-i.$$

$$\frac{2+i}{1-i} = \frac{(2+i)(1+i)}{(1-i)(1+i)} = \frac{1+3i}{1-i^2} = \frac{1+3i}{2} = \frac{1}{2} + \frac{3}{2}i.$$

Arithmetic Exercises

Simplify each expression:

1. $(1+i)+(1+3i) =$
2. $(1+2i)-5i =$
3. $i(3+i) =$
4. $(1+i)(1-i) =$
5. $(2+3i)(3+2i) =$
6. $\overline{2} =$
7. $\overline{i} =$
8. $\overline{10i} =$
9. $\overline{2+3i} =$
10. $\overline{1+5i} =$
11. $\overline{(\overline{3+i})} =$
12. $\overline{(\overline{(2-i)})} =$
13. $\overline{\overline{(1+i)}(1+i)} =$

14. $\dfrac{25}{3+4i} =$

15. $\dfrac{289+289i}{8+15i} =$

16. $\dfrac{1+i}{1-i} =$

17. $|1+i| =$

18. $|3+4i| =$

19. $|5+12| =$

Drawing Complex Numbers

Try to find out what each of the following does geometrically:

1. Multiplication by i.
2. Dividing by i.
3. Multiplication by $1 + i$.