1 Tropical Arithmetic

In tropical arithmetic, we define new addition and multiplication operations on the real numbers. The tropical sum of two numbers is their minimum:

\[ x \oplus y = \min(x, y) \]

while the tropical product of two numbers is their sum:

\[ x \odot y = x + y. \]

1. Which of the following properties hold in tropical arithmetic?

- **Addition is commutative:** \( x \oplus y = y \oplus x \).

  True. \( \min(\min(x, y), z) = \min(x, y, z) = \min(\min(y, z)) \)

- **Addition is associative:** \( (x \oplus y) \oplus z = x \oplus (y \oplus z) \).

  True. \( \min(x, y) = \min(y, x) \)

- **An additive identity exists:** There exists a real number \( n \) such that \( x \oplus n = x \) for all real numbers \( x \).

  False. Such an \( n \) would satisfy \( \min(x, n) = x \) or, equivalently, \( x \leq n \), for all real numbers \( x \).

2. Let’s expand our number set to include a tropical additive identity. What would be an appropriate name for this new “number”? Give appropriate definitions for the tropical sum and tropical product of this new number with a general real number \( x \) and with itself.

Because the tropical additive identity must be greater than or equal to every real number, we call it infinity (\( \infty \)). For any real number \( x \), we define

\[
\begin{align*}
\infty \oplus x &= x \\
\infty \oplus \infty &= \infty \\
\infty \odot x &= \infty \\
\infty \odot \infty &= \infty
\end{align*}
\]
3. Which of the following properties hold in tropical arithmetic?

- **Additive inverses exist:** For each number $x$, there exists a number $y$ such that $x \oplus y = n$, where $n$ is the additive identity.

  False. Unless $x = \infty$, there is no $y$ such that $x \oplus y = \infty$, i.e., such that $\min(x, y) = \infty$.

- **Multiplication is associative:** $(x \odot y) \odot z = x \odot (y \odot z)$.

  True. $(x + y) + z = x + (y + z)$

- **Multiplication is commutative:** $x \odot y = y \odot x$.

  True. $x + y = y + x$

- **There exists a multiplicative identity:** There exists a number $i$ such that $x \odot i = x$ for all numbers $x$.

  True. The multiplicative identity is 0: $x \odot 0 = x + 0 = x$.

- **Multiplicative inverses exist:** For each number $x$ not equal to the additive identity, there exists a number $y$ such that $x \odot y = i$, where $i$ is the multiplicative identity.

  True. For $x \neq \infty$, $x \odot (-x) = x + (-x) = 0$.

- **Multiplication distributes over addition:** $x \odot (y \oplus z) = x \odot y \odot x \odot z$.

  True. $x + \min(y, z) = \min(x + y, x + z)$

4. Complete the tropical addition and multiplication tables below.

\[
\begin{array}{c|cccc|c}
\oplus & 1 & 2 & 3 & 4 & \infty \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 2 & 2 & 2 \\
3 & 1 & 2 & 3 & 3 & 3 \\
4 & 1 & 2 & 3 & 4 & 4 \\
\infty & 1 & 2 & 3 & 4 & \infty \\
\end{array}
\quad
\begin{array}{c|cccc}
\odot & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 1 & 2 & 3 & 4 & 5 \\
3 & 1 & 2 & 3 & 4 & 5 \\
4 & 1 & 2 & 3 & 4 & 5 \\
\infty & 1 & 2 & 3 & 4 & \infty \\
\end{array}
\]
5. Expand and simplify \( f(x) = (x ⊕ 2)(x ⊕ 3) \), where juxtaposition represents tropical multiplication. Then use your expansion to compute \( f(1) \) and \( f(4) \).

\[
(x ⊕ 2)(x ⊕ 3) = x^2 ⊕ 2x ⊕ 3x ⊕ (2 ⊕ 3) \\
= x^2 ⊕ (2 ⊕ 3)x ⊕ (2 ⊕ 3) \\
= x^2 ⊕ 2x ⊕ 5
\]

\[
f(1) = 1^2 ⊕ (2 ⊕ 1) ⊕ 5 \\
= 2 ⊕ 3 ⊕ 5 \\
= 2
\]

\[
f(4) = 4^2 ⊕ (2 ⊕ 4) ⊕ 5 \\
= 8 ⊕ 6 ⊕ 5 \\
= 5
\]

2 Tropical Polynomials

A **polynomial** is an expression formed by adding and/or multiplying together numbers and copies of a variable \( x \). Every polynomial can be written in the form

\[
a_n x^n + · · · + a_2 x^2 + a_1 x + a_0
\]

for some nonnegative integer \( n \) and **coefficients** \( a_n, \ldots, a_2, a_1, a_0 \).

It follows from the **Fundamental Theorem of Algebra** that any non-constant polynomial with real coefficients can be written as a product of polynomials of degree 1 or 2 with real coefficients. For example,

\[
x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x^2 + 2x + 5)(x - 2)(x + 4)^2.
\]

Over the complex numbers, any such polynomial can be factored completely into polynomials of degree 1 with **complex coefficients**. For the example above,

\[
x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x + 1 - 2i)(x + 1 + 2i)(x - 2)(x + 4)^2.
\]

The factors can be determined by computing the **roots** (or the “zeros”) of the polynomial. The polynomial above has roots

\[-1 + 2i, -1 - 2i, 2, -4, -4.
\]

We say that the root \(-4\) has **multiplicity** 2.

There is a quadratic formula for determining the roots of a polynomial of degree 2, along with cubic and quartic formulas for degrees 3 and 4. However, starting with degree 5, there is no longer a nice
A tropical polynomial is an expression formed by (tropically) adding and/or multiplying tropical numbers (i.e., real numbers or $\infty$) and copies of a variable $x$. Every tropical polynomial can be written in the form

$$(a_n \odot x^n) \oplus \cdots \oplus (a_2 \odot x^2) \oplus (a_1 \odot x) \oplus (a_0)$$

for some nonnegative integer $n$ and coefficients $a_n, \ldots, a_2, a_1, a_0$. (Note that the exponents here represent repeated tropical multiplication.) For convenience, we represent tropical multiplication by juxtaposition, in the usual manner:

$$a_n x^n \oplus \cdots \oplus a_2 x^2 \oplus a_1 x \oplus a_0.$$

Questions:

- Can tropical polynomials always be factored completely into polynomials of degree 1?
- Is there a tropical quadratic formula for finding the roots of quadratic polynomials? How about a cubic formula?
- For polynomials of large degree, must we rely on a computer to find roots, or can we do it by hand?

### 2.1 Tropical quadratic polynomials

6. Draw a precise graph of the tropical polynomial $f(x) = x^2 \oplus 1x \oplus 4$. You may find it helpful to first rewrite the tropical polynomial (as an expression involving standard operations) using the definitions of $\oplus$ and $\odot$.

In standard notation,

$$f(x) = \min(2x, 1 + x, 4).$$

Now, try to factor the tropical polynomial $x^2 \oplus 1x \oplus 4$ into linear (degree 1) factors. In other words, find numbers $r$ and $s$ such that

$$x^2 \oplus 1x \oplus 4 = (x \oplus r)(x \oplus s).$$
These numbers $r$ and $s$ are called the **roots** of the tropical polynomial. (Note that we use $x \oplus r$ and $x \oplus s$ because we do not have a tropical subtraction.)

Because $(x \oplus r)(x \oplus s) = x^2 \oplus (r \oplus s)x \oplus rs$, we must have $r \oplus s = 1$ and $r \odot s = 4$. In standard notation, we need $\min(r, s) = 1$ and $r + s = 4$. We take $r = 1$ and $s = 3$:

$$f(x) = x^2 \oplus 1x \oplus 4 = (x \oplus 1)(x \oplus 3).$$

Do you notice any relationship between the graph and the factorization? Can you see the roots in the graph?

The roots 1 and 3 are the $x$-coordinates of the corners of the graph.

7. Graph $f(x) = -2x^2 \oplus x \oplus 8$, and then find a factorization of $f(x)$ in the form $a(x \oplus r)(x \oplus s)$. Can you see the roots $r$ and $s$ in the graph? How are the roots related to the coefficients of $f(x)$?

![Graph]

We (tropically) factor out a $-2$ to obtain

$$f(x) = -2(x^2 \oplus 2x \oplus 10).$$

Proceeding as in the previous problem, we obtain

$$f(x) = -2(x \oplus 2)(x \oplus 8).$$

The roots 2 and 8 are once again the $x$-coordinates of the corners of the graph. The roots are also the differences between consecutive coefficients of $f(x)$:

$$0 - (-2) = 2$$
$$8 - 0 = 8$$

8. Find a tropical polynomial $f(x)$ with a value of 7 for all sufficiently large $x$ and with roots 4 and 5.

We are looking for $f(x) = ax^2 \oplus bx \oplus c$. We need $f(\infty) = 7$, so the constant term $c = 7$. In view of the pattern discovered above, we subtract 5 from the value of $c$ to obtain $b = 2$, and we subtract 4 from the value of $b$ to obtain $a = -2$. We conclude that

$$f(x) = -2x^2 \oplus 2x \oplus 7.$$
Note that it does not work to subtract the roots in the other order. Indeed,

\[-2x^2 \oplus 3x \oplus 7 \neq -2(x \oplus 4)(x \oplus 5).\]

The polynomial \(-2x^2 \oplus 3x \oplus 7\) does not factor, although it defines the same function as another polynomial which does factor:

\[-2(x \oplus 4.5)^2 = -2x^2 \oplus 2.5x \oplus 7.\]

9. Graph \(f(x) = 1x^2 \oplus 3x \oplus 5\), and then find a factorization in the form \(f(x) = a(x \oplus r)(x \oplus s)\). How is this graph different from the previous ones? How is this factorization different from the others? How are the roots related to the coefficients of \(f(x)\)?

The factorization is

\[1x^2 \oplus 3x \oplus 5 = 1(x \oplus 2)^2.\]

The graphs of the three terms of \(f(x)\) intersect in a single point.

The factorization of \(f(x)\) contains a single linear factor twice, so \(f(x)\) has a root of multiplicity 2. This is explained by the fact that the differences between consecutive coefficients of \(f(x)\) are both 2.
10. Graph \( f(x) = 2x^2 \oplus 4x \oplus 4 \). Find a factorization in the form \( f(x) = a(x \oplus r)(x \oplus s) \), or show that one does not exist.

We can factor out a 2 to obtain
\[
f(x) = 2(x^2 \oplus 2x \oplus 2).
\]

However, \( x^2 \oplus 2x \oplus 2 \) does not factor. There are no \( r \) and \( s \) with minimum 2 and sum 2.

11. Can you find a tropical polynomial which has the same graph as \( f(x) = 2x^2 \oplus 4x \oplus 4 \), but which can be factored?

The polynomial \( 2x^2 \oplus 3x \oplus 4 = 2(x \oplus 1)^2 \) has the same graph as \( f(x) \).

The Tropical Fundamental Theorem of Algebra says that, for every tropical polynomial \( f(x) \), there is a unique tropical polynomial \( \bar{f}(x) \) with the same graph (and therefore determining the same function) which can be factored into linear factors. We sometimes say “the roots of \( f(x) \)” when we really mean “the roots of \( \bar{f}(x) \).”

12. If \( f(x) = ax^2 \oplus bx \oplus c \), then \( \bar{f}(x) = ax^2 \oplus Bx \oplus c \) for some \( B \). Find a formula for \( B \) in terms of \( a \), \( b \), and \( c \). There are two different cases to consider.

In order to be able to factor
\[
f(x) = a(x^2 \oplus (b - a)x \oplus (c - a)),
\]
we need to find \( r \) and \( s \) such that \( \min(r, s) = b - a \) and \( r + s = c - a \). This is possible if and only if \( 2(b - a) \leq c - a \) or, equivalently, if \( b \leq (a + c)/2 \).

Case 1: If \( b \leq (a + c)/2 \), then \( \bar{f}(x) = f(x) \) and \( B = b \).
13. State a tropical quadratic formula in terms of $a, b, c$ for the roots $x$ of a tropical polynomial $f(x) = ax^2 \oplus bx \oplus c$ (that is, the roots of the corresponding $\bar{f}$). There are once again two separate cases.

**Case 1:** If $b \leq (a + c)/2$, then $\bar{f}(x) = f(x)$ has roots $b - a$ and $c - b$, so that
\[
\bar{f}(x) = a(x \oplus (b - a))(x \oplus (c - b)).
\]

**Case 2:** If $b > (a + c)/2$, then $\bar{f}(x)$ has root $(c - a)/2$, with multiplicity 2, so that
\[
\bar{f}(x) = a \left( x \oplus \frac{c - a}{2} \right)^2.
\]

It is interesting to note that the condition $2b < a + c$ for there to be two distinct roots, when written in tropical notation, becomes $b^2 < ac$, which is reminiscent of the similar discriminant condition for standard polynomials.

### 2.2 Tropical cubic polynomials

14. For each cubic polynomial below,

- sketch the graph of the polynomial,
- use the graph to find the roots of the polynomial, and
- write (and expand) a product of linear factors with the same graph as the given polynomial.
a) $f(x) = x^3 \oplus 1x^2 \oplus 3x \oplus 6$

The roots are 1, 2, and 3, yielding a factorization

$$\bar{f}(x) = (x \oplus 1)(x \oplus 2)(x \oplus 3)$$

$$= x^3 \oplus 1x^2 \oplus 3x \oplus 6.$$

b) $g(x) = x^3 \oplus 1x^2 \oplus 6x \oplus 6$

The roots are 1, 2.5, and 2.5, yielding a factorization

$$\bar{f}(x) = (x \oplus 1)(x \oplus 2.5)^2$$

$$= x^3 \oplus 1x^2 \oplus 3.5x \oplus 6.$$
c) \( h(x) = x^3 \oplus 6x^2 \oplus 6x \oplus 6 \)

The roots are 2, 2, and 2, yielding a factorization
\[
\bar{f}(x) = (x \oplus 2)^3 = x^3 \oplus 2x^2 \oplus 4x \oplus 6.
\]

15. If \( f(x) = ax^3 \oplus bx^2 \oplus cx \oplus d \), then \( \bar{f}(x) = ax^3 \oplus Bx^2 \oplusCx \oplus d \) for some \( B \) and \( C \). With the preceding examples as a guide, find formulas for \( B \) and \( C \) in terms of \( a, b, c, \) and \( d \).

\[
B = \min \left( b, \frac{a + c}{2}, \frac{2a + d}{3} \right)
\]

\[
C = \min \left( c, \frac{b + d}{2}, \frac{a + 2d}{3} \right)
\]

2.3 General tropical polynomials

16. Can you guess the roots of the following polynomial?
\[
f(x) = 3x^6 \oplus 4x^5 \oplus 2x^4 \oplus x^3 \oplus 1x^2 \oplus 4x \oplus 5
\]

We have
\[
\bar{f}(x) = 3x^6 \oplus 2x^5 \oplus 1x^4 \oplus x^3 \oplus 1x^2 \oplus 3x \oplus 5,
\]
so the roots are \(-1, -1, -1, 1, 2, 2\).

17. If
\[
f(x) = a_n x^n \oplus a_{n-1} x^{n-1} \oplus \cdots \oplus a_2 x^2 \oplus a_1 x \oplus a_0,
\]
then
\[
\bar{f}(x) = a_n x^n \oplus A_{n-1} x^{n-1} \oplus \cdots \oplus A_2 x^2 \oplus A_1 x \oplus a_0.
\]
Can you find a formula for each $A_j$ in terms of the $a_i$?

$$A_j = \min_{l \leq j < k} \left( \frac{a_l - a_k}{k - l} (k - j) + a_k \right)$$
$$= \min_{l \leq j < k} \left( a_l \left( \frac{k - j}{k - l} \right) + a_k \left( \frac{j - l}{k - l} \right) \right),$$

an appropriately weighted average of some $a_l$ and $a_k$, with $l \leq j < k$.

How about formulas for the roots $r_1, r_2, \ldots, r_n$?

The roots are simply the differences between consecutive coefficients of $\bar{f}(x)$. That is,

$$r_i = A_i - A_{i-1}$$

(where we set $A_n = a_n$ and $A_0 = a_0$).

Can you find a geometric interpretation of these formulas in terms of the points $(-i, a_i)$, for $0 \leq i \leq n$?

The inequality

$$A_j \leq \frac{a_l - a_k}{k - l} (k - j) + a_k$$

(for $l \leq j < k$) states that the point $(-j, A_j)$ must lie on or below the line segment between the points $(-k, a_k)$ and $(-l, a_l)$. This makes it easy to find the $A_j$ using a graph of the points $(-i, a_i)$ for $0 \leq i \leq n$. 