Metrics 2

Definition: Let X be any nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** if

i) $d(x, y) \geq 0$ for all $x, y \in X$.

ii) For $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

iii) $d(x, y) = d(y, x)$ for any $x, y \in X$.

iv) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

The inequality in (iv) is known as the **triangle inequality**.

A set X equipped with a metric d is called a **metric space**, denoted (X, d).

Last time, we saw two metrics: the Euclidean metric and the Taxicab metric on $X = \mathbb{R}^n$. For $x = (x_1, ..., x_n) \in R^n$ and $y = (y_1, ..., y_n) \in R^n$, we defined

$$d_E(x, y) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$$

$$d_T(x, y) = |x_1 - y_1| + ... + |x_n - y_n|$$

We now investigate other metrics, as well as properties of metric spaces in general.

Problem 1: Define the **discrete metric** on any nonempty set X as follows:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Show d is indeed a metric.

Problem 2: Let X be the set of English words with 6 letters. Define the **Hamming distance** between two strings as the number of positions in which the two strings differ. For example, you can't compare apples and oranges, but $d(apples, orange) = 6$. Meanwhile, $d(zigzag, puzzle) = 5$. Verify the Hamming distance gives a metric on X. (If you're bored, come up with some 6 letter words).

Problem 3:

a) Let $X = \mathbb{N}$ be the set of natural numbers (positive integers). Define $d(a, b) = \left|\frac{a}{b} - \frac{b}{a}\right|$. Is d a metric on $\mathbb{N}$?

b) Let X be a nonempty set and $\mathcal{P}(X)$ its power set, i.e. the set of all subsets of X. For two subsets $A, B \subset X$, define $d(A, B) = |(A \cup B) \setminus (A \cap B)|$. Is d a metric on $\mathcal{P}(X)$?

Problem 4: Let $X = \mathbb{R}^n$.

a) For $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$, define

$$d(x, y) = |x_1 - y_1|$$

Is d a metric on $\mathbb{R}^n$?

b) For $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$, define

$$d(x, y) = \max(|x_1 - y_1|, ..., |x_n - y_n|)$$

Is d a metric on $\mathbb{R}^n$?

c) For $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$, define

$$d(x, y) = \min(|x_1 - y_1|, ..., |x_n - y_n|)$$

Is d a metric on $\mathbb{R}^n$?

Definition: Let (X, d) be a metric space. An **open ball** of radius $r > 0$ centered at a point $x_0 \in X$ is the set $B(x_0, r) := \{x \in X : d(x, x_0) < r\}$.

Problem 5:

a) Let $X = \mathbb{R}^2$ be equipped with the standard Euclidean metric d_E. Draw the open ball $B((0, 0), 1)$.

b) Let $X = \mathbb{R}^1$ be equipped with the standard Euclidean metric d_E. Draw the open ball $B(0, 1)$.

c) What do open balls look like in $X = \mathbb{R}^3$ equipped with the standard Euclidean metric?

d) Let $X = \mathbb{R}^2$ be equipped with the taxicab metric d_T. Draw the open ball $B((0, 0), 1)$. What do open balls in $\mathbb{R}^3$ with the taxicab metric look like?

Problem 6: Describe (via picture, list, or other suitable description) the following open balls in the given metric space.

a) $B((0,0),1)$ in $X=\mathbb{R}^2$ equipped with the metric from Problem 4b.

b) $B(0,1)$ in $X=\mathbb{R}^2$ equipped with the discrete metric (from Problem 1).

c) $B(0,2)$ in $X=\mathbb{R}^2$ equipped with the discrete metric

d) $B(cats,1)$ in $X=$ the set of 4 letter English words equipped with Hamming distance (from Problem 2).

e) $B(dogs,2)$ with the same metric space as 6d

f) $B(\{1,2,3\},2)$, in the metric space of problem $3b$ on $X=\{1,2,3,4\}$.

Definition: Let (X, d) be a metric space. We say an infinite sequence $x_1, x_2, x_3, x_4, ...$ of elements of X *converges* to an element $x \in X$ if every open ball centered at x can only avoid finitely many elements in the sequence (and contains the rest).

Example: Let $X = \mathbb{R}$ with the standard Euclidean metric, i.e. $d_E(x, y) = |x - y|$. The sequence of real numbers $1/1, 1/2, 1/3, 1/4, 1/5, 1/6, ...$ converges to 0. (We may write this sequence as $(1/n)_{n=1}^{\infty}$).

To see this sequence indeed converges to 0, we need to check every open ball in $\mathbb{R}$ centered at 0 can only avoid finitely many numbers on our list. For instance, let's take the open ball centered at 0 of radius 1/2. We have

$$B(0, 1/2) = (-1/2, 1/2) = \{x \in \mathbb{R} : -1/2 < x < 1/2\}$$

is just an open interval. It does not contain 1/1 or 1/2. However, it does contain $1/3, 1/4, 1/5, ...$ More generally, if we have any radius $r > 0$,

$$B(0, r) = (-r, r) = \{x \in \mathbb{R} : -r < x < r\}$$

Notice that since $1/n$ is always positive for $n = 1, 2, 3, ...$, it will be in this interval if and only if $1/n < r$. Equivalently, we need $n > 1/r$. So, while $1/1, 1/2, ..., 1/(\lceil 1/r \rceil - 1)$ are not in this ball, $1/(\lceil r \rceil), 1/(\lceil r \rceil + 1), ...$ are. This shows that a ball centered around 0 of any radius only avoids finitely many numbers in our sequence. So the sequence converges to 0.

Problem 7:

a) By the example above, it's true in the Euclidean metric on $\mathbb{R}$ that $(1/n)_{n=1}^{\infty}$ converges to 0. Does the sequence $(1/2^m)_{m=1}^{\infty}$ (i.e. the sequence $1/2, 1/4, 1/8, 1/16, ...$) in $\mathbb{R}$ with the Euclidean metric converge to anything? What can you say about *subsequences* of a sequence that converges?

b) On the other hand, consider $\mathbb{R}$ with the discrete metric. What do the open balls $B(0, r)$ look like in this case? (Consider $r \leq 1$ and $r > 1$). Does $(1/n)_{n=1}^{\infty}$ converge to 0? Why or why not? Does it converge to anything other than 0?

c) Can a sequence converge to more than one point? Provide a proof.

Problem 8: Consider the metric space from 6d.

a) Does the sequence $cats, cats, cats, cats, ...$ converge to anything?

b) Does the sequence $dogs, cats, cats, cats, ...$ converge to anything?

c) Does the sequence $hats, cats, hats, cats, hats, cats, ...$ converge to anything?

d) In order for a sequence of 4 letter words to converge in this metric space, what must be true?

Problem 9: While the taxicab metric and the Euclidean metric are not the same, it turns out any sequence $x_1, x_2, x_3, ...$ converges to $x \in \mathbb{R}^n$ with respect to the taxicab metric if and only if it converges to x with respect to the Euclidean metric.

a) Show geometrically that any Euclidean open ball in $\mathbb{R}^2$ contains a Taxicab open ball with the same center. If the Euclidean open ball has radius 1, what should the taxicab open ball radius be?

b) Conversely, show any Taxicab open ball in $\mathbb{R}^2$ contains a Euclidean open ball with the same center. If the Taxicab open ball has radius 1, what should the Euclidean open ball radius be?

c) Conclude that a sequence $x_1, x_2, x_3, x_4, ...$ converges to $x \in \mathbb{R}^2$ in the Euclidean metric if and only if the sequence converges to x in the Taxicab metric. *Hint*: Suppose the sequence didn't converge to x in the Taxicab metric. Then we could find some Taxicab open ball centered at x avoiding infinitely many elements of our sequence. Could we find a Euclidean open ball with the same property?

Remark: Metrics satisfying the corresponding condition to 9a and 9b are called *equivalent*. As far as sequences are concerned, such metrics are no different.

Challenge Section - Strongly Equivalent Metrics

Definition: Two metrics d_1, d_2 on a space X are said to be *strongly equivalent* if there exist constants $a, b > 0$ with

$$a \cdot d_1(x, y) \leq d_2(x, y) \leq b \cdot d_1(x, y)$$

for each $x, y \in X$.

1. Show the taxicab metric and the Euclidean metric on $\mathbb{R}^2$ are strongly equivalent. (*Hint*: Taking $d_1 = d_T, d_2 = d_E$ in the above, take $a = 1/3$).

2. In general, show the taxicab and Euclidean metric on $\mathbb{R}^n$ are strongly equivalent.

3. Let d_1, d_2 be two strongly equivalent metrics on X. Show that d_1 and d_2 are equivalent. (It should be easy to relate radii now). This gives an alternative solution to Problem 9.

4. Let (X, d_1) be a metric space. Define $d_2 : X \times X \to \mathbb{R}$ via $d_2(x, y) = \frac{d_1(x,y)}{1+d_1(x,y)}$. Show d_2 is a metric. Is it in general strongly equivalent to d_1? Is it equivalent?

5. Let (X, d) be a metric space. Define $d_3 : X \times X \to \mathbb{R}$ via $d_3(x, y) = \min(d_1(x, y), 1)$. Show d_3 is a metric. Is it in general equivalent or strongly equivalent to d_1 or d_2?

Challenge Section - Weird Metric Spaces

1. a) Let $G = (V, E)$ be an undirected graph. For $v, w \in V$, define $d_G(v, w)$ to be the length of the shortest path from v to w in the graph G. Is d_G a metric?

   b) Let $V = \{000, 001, 010, 011, 100, 101, 110, 111\}$. Draw a graph $G = (V, E)$ with this vertex set such that the graph distance, d_G, is the same as the Hamming distance.

   c) What do open balls look like in the previous problem?

2. Let X_0 be the set of infinite sequences of real numbers, $(a_n)_{n=1}^{\infty}$, which are eventually constant at 0, i.e. have some $N \in \mathbb{N}$ such that $a_n = 0$ for all $n \geq N$. Let $d_1((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} |a_n - b_n|$. Note that this is only a finite sum since our sequences are eventually 0. Show d_1 is a metric on X_0.

3. Similarly, let $d_\infty((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}) = \max_{n=1}^{\infty} |a_n - b_n|$. Notice all but finitely many of the $a_n - b_n$ are just $0 - 0 = 0$, so it makes sense to take the maximum. Show d_∞ is a metric on X_0.

4. Construct a sequence in X_0 which converges to something in the d_∞ metric but not the d_1 metric. (Note that this is a sequence of elements in a set X_0, but the elements of X_0 are themselves sequences).

Challenge Section - Cauchy Sequences

A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is called *Cauchy* if, given any radius $r > 0$ (however small), we can center a ball of radius r somewhere so that all but finitely many of the elements of the sequence are in this ball. (No matter how small, there is a ball that contains almost all.)

1. Draw a Cauchy sequence in $\mathbb{R}^2$ (with the Euclidean metric).

2. Show that any convergent sequence (in any metric space) is also a Cauchy sequence.

3. Give an example of a metric space with a Cauchy sequence which does not converge to anything.

A metric space where every Cauchy sequence also happens to converge is called *complete*. It turns out $\mathbb{R}^n$ with the Euclidean metric is complete.