Challenge Section - Geometry

1. The (Euclidean) ellipse with foci at points $p, q \in \mathbb{R}^2$ and parameter $b > d_E(p,q)$ can be defined as the set
$$\{x \in \mathbb{R}^2 : d_E(x,p) + d_E(x,q) = b\}$$
In a similar setting, describe (via sketch) the taxicab ellipse
$$\{x \in \mathbb{R}^2 : d_T(x,p) + d_T(x,q) = b\}$$
Consider, in particular, the special cases $p = (1,0)$, $q = (-1,0)$, and $p = (1,1)$, $q = (-1,-1)$.

2. a) In Euclidean geometry, the perpendicular bisector between two points p, q may be written as $\{x \in \mathbb{R}^2 : d_E(x,p) = d_E(x,q)\}$. Draw the Taxicab version $\{d_T(x,p) = d_T(x,q)\}$. Consider, in particular, the special cases $p = (1,0)$, $q = (-1,0)$, and $p = (1,1)$, $q = (-1,-1)$.

   b) In Euclidean geometry, the hyperbola with foci p, q and parameter b may be written as $\{x \in \mathbb{R}^2 : d_E(x,p) = d_E(x,q) + b\}$. Draw the taxicab version, $\{x \in \mathbb{R}^2 : d_T(x,p) = d_T(x,q) + b\}$. Consider, in particular, the special cases $p = (1,0)$, $q = (-1,0)$, and $p = (1,1)$, $q = (-1,-1)$.

3. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is the translation on $\mathbb{R}^n$ sending 0 to x_0 and shifting everything else accordingly,we have a simple formula for T given by $T(x) = x + x_0$. It is easy to see that d_E and d_T are *translation invariant*. That is, for any points $x, y \in \mathbb{R}^n$, we have
$$d_E(T(x), T(y)) = d_E(x,y)$$
$$d_T(T(x), T(y)) = d_T(x,y)$$
One may geometrically reason that d_E is rotation-invariant. Draw a picture to see geometrically why this ought to be true. Show via example (in $\mathbb{R}^2$) that d_T is *not* rotation-invariant. That is, find two points, and rotate them about the origin by a certain angle. See that the distance before the rotation and the distance after the rotation are different.

Challenge Section - Proving d_E is a metric

Definition: Define the *dot product* on $\mathbb{R}^n$ to be a map $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by
$$\langle (x_1, ..., x_n), (y_1, ..., y_n) \rangle = x_1 y_1 + ... + x_n y_n$$
Observe for any $x \in \mathbb{R}^n$, $d_E(x,0)^2 = \langle x, x \rangle$.

1. Prove the *Cauchy-Schwarz inequality*:
$$|\langle x, y \rangle| \leq d_E(x,0) \cdot d_E(y,0)$$
Hint: For $y = 0$ there is nothing to show. Suppose $y \neq 0$. Let $c = \langle x, y \rangle / d_E(y,0)^2$. Compute $d_E(x - c \cdot y, 0)^2$, which we know to be positive.

2. When is there equality in the Cauchy-Schwarz inequality? (Revisit your proof).

3. Prove (using the previous problem) the triangle inequality holds for the Euclidean metric on $\mathbb{R}^n$. Use this to complete the proof we skipped earlier that d_E is a metric on $\mathbb{R}^n$.