Challenge Section - Geometry

1. The (Euclidean) ellipse with foci at points \( p, q \in \mathbb{R}^2 \) and parameter \( b > d_E(p, q) \) can be defined as the set

\[
\{ x \in \mathbb{R}^2 : d_E(x, p) + d_E(x, q) = b \}
\]

In a similar setting, describe (via sketch) the taxicab ellipse

\[
\{ x \in \mathbb{R}^2 : d_T(x, p) + d_T(x, q) = b \}
\]

Consider, in particular, the special cases \( p = (1, 0) \), \( q = (-1, 0) \), and \( p = (1, 1) \), \( q = (-1, -1) \).

2. a) In Euclidean geometry, the perpendicular bisector between two points \( p, q \) may be written as \( \{ x \in \mathbb{R}^2 : d_E(x, p) = d_E(x, q) \} \). Draw the Taxicab version \( \{ d_T(x, p) = d_T(x, q) \} \). Consider, in particular, the special cases \( p = (1, 0) \), \( q = (-1, 0) \), and \( p = (1, 1) \), \( q = (-1, -1) \).

b) In Euclidean geometry, the hyperbola with foci \( p, q \) and parameter \( b \) may be written as \( \{ x \in \mathbb{R}^2 : d_E(x, p) = d_E(x, q) + b \} \). Draw the taxicab version, \( \{ x \in \mathbb{R}^2 : d_T(x, p) = d_T(x, q) + b \} \). Consider, in particular, the special cases \( p = (1, 0) \), \( q = (-1, 0) \), and \( p = (1, 1) \), \( q = (-1, -1) \).

3. If \( T : \mathbb{R}^n \to \mathbb{R}^n \) is the translation on \( \mathbb{R}^n \) sending \( 0 \) to \( x_0 \) and shifting everything else accordingly, we have a simple formula for \( T \) given by \( T(x) = x + x_0 \). It is easy to see that \( d_E \) and \( d_T \) are translation invariant. That is, for any points \( x, y \in \mathbb{R}^n \), we have

\[
d_E(T(x), T(y)) = d_E(x, y)
\]

\[
d_T(T(x), T(y)) = d_T(x, y)
\]

One may geometrically reason that \( d_E \) is rotation-invariant. Draw a picture to see geometrically why this ought to be true. Show via example (in \( \mathbb{R}^2 \)) that \( d_T \) is not rotation-invariant. That is, find two points, and rotate them about the origin by a certain angle. See that the distance before the rotation and the distance after the rotation are different.

Challenge Section - Proving \( d_E \) is a metric

Definition: Define the dot product on \( \mathbb{R}^n \) to be a map \( \langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) given by

\[
\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = x_1y_1 + \ldots + x_ny_n
\]

Observe for any \( x \in \mathbb{R}^n \), \( d_E(x, 0)^2 = \langle x, x \rangle \).

1. Prove the Cauchy-Schwarz inequality:

\[
|\langle x, y \rangle| \leq d_E(x, 0) \cdot d_E(y, 0)
\]

Hint: For \( y = 0 \) there is nothing to show. Suppose \( y \neq 0 \). Let \( c = \langle x, y \rangle / d_E(y, 0)^2 \). Compute \( d_E(x - c \cdot y, 0)^2 \), which we know to be positive.

2. When is there equality in the Cauchy-Schwarz inequality? (Revisit your proof).

3. Prove (using the previous problem) the triangle inequality holds for the Euclidean metric on \( \mathbb{R}^n \). Use this to complete the proof we skipped earlier that \( d_E \) is a metric on \( \mathbb{R}^n \).