Take-Away Games

Introduction

Today we will play and study games.

- Every game will be played by two *players:* Player I and Player II.

- A game starts with a certain *position* and follows some *rules.*

- Players take turns making *moves*. Player I moves first.

- Every game has a *goal.* The player who achieves the goal first *wins* the game.

In some of the games you are familiar with winning depends on luck. In other games, you can come up with a *strategy* that allows you to win no matter what your opponent does. In some of the games, there is no such strategy (or at least such a strategy is not known).

The type of games we will play today are called *Take-Away Games.*

- Initial position is a pile (or a row) of chips;

- A move consists of taking several chips (players are allowed to take only specific numbers of chips on each move);

- The goal of the game is to take the last chip;

- The player who takes the last chip wins the game.
  We will play several take-away games with the goal of finding a *winning strategy* for each such game. That is, we want to find a method (for one of the players) to win, no matter what their opponent does.

Take one!

Rules: Two players take turns taking away 1 chip at a time from a pile of chips. The player who takes the last chip wins.

Starting Position: a pile of several chips;
Move: take 1 chip;
Goal: take the last chip;

Play this game several time. As you are playing, fill out the table below. Write down the total number of chips you start with on the left and who won (Player I or Player II) on the right.

Total numbers of Chips	Winner (Player I or Player II)

Can you predict which player will win depending how many chips you started with? Without playing the game, fill out the table below:

Total numbers of Chips	Winner (Player I or Player II)
5	
6	
12	
17	
100	
2017	

Complete the sentences:

- If the total number of chips is _____, Player I wins.
- If the total number of chips is _____, Player II wins.

Do any of the players have to make any decisions as they play this game? Why or why not?

Take one or two!

Let's see what happens when the rules get more complicated.

*Rules:*Two players take turns taking away 1 or 2 chips at a time from a pile of chips. The player who takes the last chip wins.

Starting Position: a pile of several chips;
Move: take 1 or 2 chips (your choice!);
Goal: take the last chip;

Discussion Questions:

1. Why is this game more complicated than the game *Take One* is?

2. What decision does each player have to make on every move?

We will play this game several times starting with different numbers of chips.

1. Start with 2 chips. Which player (Player I or Player II) can win this game?

2. Start with 3 chips. Play this game with a partner several times.

   (a) Which of the players (Player I or Player II) has a strategy to win this game (no matter how the other player plays)?

   (b) Describe the winning strategy for the starting position with 3 chips.

3. Start with 4 chips. Play this game with a partner several times.

(a) Which of the players (Player I or Player II) has a strategy to win this game?

(b) Describe the winning strategy for the starting position with 4 chips.

Strategy or not?

Alice, Bob, Cindy, David, Evelyn and Frank are going to a party where they will be playing *Take Away games* with each other. In the game they will play, players are allowed to take away any number of chips between 1 and 5.

The children plan ahead how they will play. For each of their ideas, describe if it forms a *strategy* in the mathematical sense. You do not need to decide if it is a good strategy or a bad strategy, just whether their idea describes a strategy at all.

1. Alice says that she will always take away exactly 1 chip, as long as there are chips available. Is this a strategy? Please circle Yes or No.

2. Bob says that he will sometimes take 1 chip and sometimes will take 3 chips (because this is his favorite number).
   Is this a strategy? Please circle Yes or No.

3. Cindy says that she will close her eyes and take however many chips she can grab without looking.
   Is this a strategy? Please circle Yes or No.

4. David says that he will take 3 chips on the first move, and for the rests of the moves, he will take 1 chip every time.
   Is this a strategy? Please circle Yes or No.

5. Evelyn says that she will start by taking 1 chip and then will 2 chips for as long as there are over 5 chips left. Once there are fewer than 5 chips left, she will take 1 chip away each time.
   Is this a strategy? Please circle Yes or No.

6. Frank says that he will always take fewer chips than his opponent takes on the previous move.
   Is this a strategy? Please circle Yes or No.

Winning and Losing Positions

We will call a position in Take-Away game a *Winning Position* (or simply *W)* if the player who has to move from this position has a winning strategy.

We will call a position in a Take-Away a *Losing Position* if the opponent of the player who has to move in this position has a winning strategy. This means that the player who starts moving from this position will lose (if their opponent follows the winning strategy).

1. For the game *Take One!,* mark the Winning positions by W and the Losing positions by L on the number line below.

2. For the game *Take One or Two!,* mark the Winning positions by W and the Losing positions by L on the number line below.

What is the pattern?

Winning strategies for both Players?

Alice and Bob play a take-away game for a long time. Then Alice announces that she always wants to be Player I because she found a winning strategy for Player I. Bob says that he always wants to be Player II because he found a winning strategy for Player II.

Is this possible? Can both players (Player I and Player II) have a winning strategy for the same take-away game?

Take one or three!

Starting Position: a pile of several chips;

Move: take either 1 or 3 chips;

Goal: take the last chip;

1. Each number on the number line below represents the number of chips in the beginning of the game. For each starting position, decide whether it is a winning or a losing position and write a W or an L above the corresponding number on the number line.

   (a) What is the pattern?

   (b) Which player has a winning strategy if the game starts with 100 chips?

Make your own game

Time for some fun!

Come up with your own take away game. Choose a number (e.g., 5). Play the *take-away game* in which you are allowed to take either 1 chip or the chosen number of chips (in our example, 5). Write down the rules of your game. Then play against three of your classmates who have already finished their work.

Challenge: Do not take the Last Chip!

The rules have now changed! Two people are playing a game where there is a pile of chips. Players I and II take turns taking chips from the pile. The player who is forced to take the last chip **loses.**

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

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