Metrics 1

Definition: The **Euclidean distance** between two points $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, ..., y_n) \in \mathbb{R}^n$ may be computed as

$$d_E(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$$

In practice, straight line distance may not be the most relevant notion of distance. For instance, if you are traveling in a city, you are constrained: you can only walk along the sidewalks, which either go north and south or east and west.

Problem 1: This introduces a more relevant notion of distance between points in $\mathbb{R}^2$.

a) For $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$, write down a formula for $d_T(x, y)$, the **Taxicab distance** between these two points.

$d_T(x, y) :=$ ______________________________

b) Distance should always be nonnegative. Verify the above definition indeed ensures $d_T(x, y) \geq 0$ for each $x, y \in \mathbb{R}^2$. Find (with proof) a necessary and sufficient condition for when $d_T(x, y) = 0$.

c) While the analogy of taxicabs breaks down in higher dimensions, for $x = (x_1, ..., x_n) \in \mathbb{R}^n, y = (y_1, ..., y_n) \in \mathbb{R}^n$, write down an analogous definition for $d_T(x, y)$. Does it have a similar property as in 1b?

$d_T(x, y) :=$ ______________________________

Problem 2: Compute $d_E(x, y)$ and $d_T(x, y)$ for the following points $x, y \in \mathbb{R}^n$ for some n.

a) $x = (1, 1, 3, 4), y = (-3, 7, -1, 0) \in \mathbb{R}^4$

b) $x = (-3, 7, -1, 0), y = (1, 1, 3, 4) \in \mathbb{R}^4$

c) Prove $d_E(x, y) = d_E(y, x)$ and $d_T(x, y) = d_T(y, x)$ for any $x, y \in \mathbb{R}^n$. Our notions of distance are *symmetric*.

Problem 3:

a) (Optional) Prove by induction the following two *power sum formulas*.

$$\sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

b) Compute and simplify the expressions for $d_E(x, y)$ and $d_T(x, y)$ for $x = (0, ..., 0), y = (1, 2, 3, ..., n) \in \mathbb{R}^n$.

c) In part 3*b* above, which is larger: $d_E(x, y)$ or $d_T(x, y)$? *Hint*: It may help to compare $d_E(x, y)^2$ and $d_T(x, y)^2$ instead.

Problem 4: Let $x, y \in \mathbb{R}^n$ be two arbitrary points.

a) Which paths between them corresponds to the notion of distance $d_E(x, y)$? Which paths between them correspond to the notion of distance $d_T(x, y)$? It might help to draw a picture.

b) For $n = 1$, show $d_E(x, y) = d_T(x, y)$ for each $x, y \in \mathbb{R}^1$.

c) For $n = 2$, show $d_E(x, y) \leq d_T(x, y)$ for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Find (with proof) necessary and sufficient conditions on x_1, x_2, y_1, y_2 for there to be equality. It may help to refer to your picture in part 4a.

Remark: A similar result (with similar proof) holds for arbitrary n. You can try to prove it in the general case, if you'd like.

Problem 5: (Taxicab Shapes) The (Euclidean) unit circle can be written as the subset $\{x \in \mathbb{R}^2 : d_E(x, 0) = 1\}$. The (Euclidean) line segment between two points $p, q \in \mathbb{R}^n$ can be written as $\{x \in \mathbb{R}^n : d_E(p, x) + d_E(x, q) = d_E(p, q)\}$.

a) Draw the Taxicab unit circle: $\{x \in \mathbb{R}^2 : d_T(x, 0) = 1\}$.

b) Draw the Taxicab line segment between two points $p, q \in \mathbb{R}^2$: $\{x \in \mathbb{R}^2 : d_T(p, x) + d_T(x, q) = d_T(p, q)\}$.

Problem 6: While points x, y, z (in order) on a (Euclidean) line segment satisfy $d_E(x, y) + d_E(y, z) = d_E(x, y)$, if y is not on the line segment between x and z, we satisfy the triangle inequality: $d_E(x, z) \leq d_E(x, y) + d_E(y, z)$.

a) Show $|a + b| \leq |a| + |b|$ for $a, b \in \mathbb{R}$. Use this to conclude the triangle inequality holds for the Euclidean metric in $\mathbb{R}^1$. That is, show

$$d_E(x, z) \leq d_E(x, y) + d_E(y, z) \text{ for any } x, y, z \in \mathbb{R}^1$$

(The case for $\mathbb{R}^n$ is harder.)

b) Show the taxicab metric satisfies $d_T(x, z) \leq d_T(x, y) + d_T(y, z)$ for any $x, y, z \in \mathbb{R}^n$. (Try $n = 1, 2$ first if you are not sure how to start).

Challenge Section - Geometry

1. The (Euclidean) ellipse with foci at points $p, q \in \mathbb{R}^2$ and parameter $b > d_E(p, q)$ can be defined as the set
$$\{x \in \mathbb{R}^2 : d_E(x, p) + d_E(x, q) = b\}$$
In a similar setting, describe (via sketch) the taxicab ellipse
$$\{x \in \mathbb{R}^2 : d_T(x, p) + d_T(x, q) = b\}$$
Consider, in particular, the special cases $p = (1, 0)$, $q = (-1, 0)$, and $p = (1, 1)$, $q = (-1, -1)$.

2. a) In Euclidean geometry, the perpendicular bisector between two points p, q may be written as $\{x \in \mathbb{R}^2 : d_E(x, p) = d_E(x, q)\}$. Draw the Taxicab version $\{x \in \mathbb{R}^2 : d_T(x, p) = d_T(x, q)\}$. Consider, in particular, the special cases $p = (1, 0)$, $q = (-1, 0)$, and $p = (1, 1)$, $q = (-1, -1)$.

   b) In Euclidean geometry, the hyperbola with foci p, q and parameter b may be written as $\{x \in \mathbb{R}^2 : d_E(x, p) = d_E(x, q) + b\}$. Draw the taxicab version, $\{x \in \mathbb{R}^2 : d_T(x, p) = d_T(x, q) + b\}$. Consider, in particular, the special cases $p = (1, 0)$, $q = (-1, 0)$, and $p = (1, 1)$, $q = (-1, -1)$.

3. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is the translation on $\mathbb{R}^n$ sending 0 to x_0 and shifting everything else accordingly,we have a simple formula for T given by $T(x) = x + x_0$. It is easy to see that d_E and d_T are *translation invariant*. That is, for any points $x, y \in \mathbb{R}^n$, we have
$$d_E(T(x), T(y)) = d_E(x, y)$$
$$d_T(T(x), T(y)) = d_T(x, y)$$
One may geometrically reason that d_E is rotation-invariant. Draw a picture to see geometrically why this ought to be true. Show via example (in $\mathbb{R}^2$) that d_T is *not* rotation-invariant. That is, find two points, and rotate them about the origin by a certain angle. See that the distance before the rotation and the distance after the rotation are different.

Challenge Section - Proving d_E is a metric

Definition: Define the *dot product* on $\mathbb{R}^n$ to be a map $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by
$$\langle (x_1, ..., x_n), (y_1, ..., y_n) \rangle = x_1 y_1 + ... + x_n y_n$$
Observe for any $x \in \mathbb{R}^n$, $d_E(x, 0)^2 = \langle x, x \rangle$.

1. Prove the *Cauchy-Schwarz inequality*:
$$|\langle x, y \rangle| \leq d_E(x, 0) \cdot d_E(y, 0)$$
Hint: For $y = 0$ there is nothing to show. Suppose $y \neq 0$. Let $c = \langle x, y \rangle / d_E(y, 0)^2$. Compute $d_E(x - c \cdot y, 0)^2$, which we know to be positive.

2. When is there equality in the Cauchy-Schwarz inequality? (Revisit your proof).

3. Prove (using the previous problem) the triangle inequality holds for the Euclidean metric on $\mathbb{R}^n$. Use this to complete the proof we skipped earlier that d_E is a metric on $\mathbb{R}^n$.