Theorem (Ceva). Let $\triangle ABC$ be a triangle, and $D, E, F$ points inside the line segments $BC, CA, AB$ respectively. Then $AD, BE, CF$ are concurrent if and only if
\[
\frac{AF}{BD} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = 1.
\]
Note: There is a more general result for when $D, E, F$ are outside. But then one must use directed lengths!

Problem 1. For a positive integer $n$, define $\mu(n)$ to be $(-1)^k$ if $n$ is of the form $p_1 \cdots p_k$ where $p_1, \ldots, p_k$ are distinct primes, and 0 otherwise. So for example $\mu(1) = 1$, $\mu(2) = -1$, $\mu(4) = 0$, $\mu(6) = 1$.

(a) Show that if $p_1, \ldots, p_k$ are distinct primes ($k \geq 1$), then
\[
\sum_{d \mid p_1 \cdots p_k} \mu(d) = 0,
\]
where the sum is over the positive divisors of the product $p_1 \cdots p_k$. \textit{Hint: Induct on $k$.}

(b) Show that in fact for any integer $n \geq 2$,
\[
\sum_{d \mid n} \mu(d) = 0,
\]
where the sum is over the positive divisors of $n$.

Problem 2. Let $\triangle ABC$ be an acute triangle, $M, N, P$ the midpoints of $BC, CA, AB$, and $D, E, F$ the feet of the perpendiculars from $A, B, C$. Let $M', N', P'$ be the midpoints of $AD, BE, CF$ respectively. Show that $MM', NN', PP'$ are concurrent. \textit{Hint: Ceva twice.}

Problem 3. Suppose that a $10 \times 10$ square can be covered by $N$ disks of radius 2. Show that the same $10 \times 10$ square can be covered by $4N$ disks of radius 1.

Problem 4. Let $n$ be a positive integer. How many ways are there to write $2^n \cdot 13$ as a sum of two squares $x^2 + y^2$, where $x$ and $y$ are positive integers?

Problem 5. (a) Are there any real $a, b, c$ such that $a + b + c = 7$ and $ab + bc + ca = 17$?

(b) Are there any real $a, b, c$ such that $a + b + c = 7$, $ab + bc + ca = 16$ and $abc = 13$?

\textit{Hint: Try to reach a contradiction by breaking some well-known inequalities.}