PROBLEM SESSION 1

LAMC OLYMPIAD GROUP, WEEK 1

Theorem (Ceva). Let $\triangle ABC$ be a triangle, and D, E, F points inside the line segments BC, CA, AB respectively. Then AD, BE, CF are concurrent *if and only if*

$$\frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = 1.$$

Note: There is a more general result for when D, E, F are outside. But then one must use directed lengths!

Problem 1. For a positive integer n, define $\mu(n)$ to be $(-1)^k$ if n is of the form $p_1 \cdots p_k$ where $p_1, \ldots, p_k$ are distinct primes, and 0 otherwise. So for example $\mu(1) = 1$, $\mu(2) = -1$, $\mu(4) = 0$, $\mu(6) = 1$.

(a) Show that if $p_1, \ldots, p_k$ are distinct primes ($k \geq 1$), then

$$\sum_{d \mid p_1 \cdots p_k} \mu(d) = 0,$$

where the sum is over the positive divisors of the product $p_1 \cdots p_k$. *Hint: Induct on k.*

(b) Show that in fact for any integer $n \geq 2$,

$$\sum_{d \mid n} \mu(d) = 0,$$

where the sum is over the positive divisors of n.

Problem 2. Let $\triangle ABC$ be an acute triangle, M, N, P the midpoints of BC, CA, AB, and D, E, F the feet of the perpendiculars from A, B, C. Let M', N', P' be the midpoints of AD, BE, CF respectively. Show that MM', NN', PP' are concurrent. *Hint: Ceva twice.*

Problem 3. Suppose that a 10×10 square can be covered by N disks of radius 2. Show that the same 10×10 square can be covered by $4N$ disks of radius 1.

Problem 4. Let n be a positive integer. How many ways are there to write $2^n \cdot 13$ as a sum of two squares $x^2 + y^2$, where x and y are positive integers?

Problem 5. **(a)** Are there any real a, b, c such that $a + b + c = 7$ and $ab + bc + ca = 17$?
(b) Are there any real a, b, c such that $a + b + c = 7$, $ab + bc + ca = 16$ and $abc = 13$?
Hint: Try to reach a contradiction by breaking some well-known inequalities.