OLYMPIAD-STYLE PROBLEMS II
COLLECTED FOR THE
LOS ANGELES MATH CIRCLE

Problem 1 (2010 AIME II Problem 2 ©MAA).
A point $P$ is chosen at random in the interior of a unit square $S$. Let $d(P)$ denote the distance from $P$ to the closest side of $S$. Find the probability that $1/5 \leq d(P) \leq 1/3$.

Solution: The event in question corresponds to the shaded region in the following picture.

The innermost square has vertices $(1/3, 1/3), (1/3, 2/3), (2/3, 2/3)$, and $(2/3, 1/3)$. The middle square has vertices $(1/5, 1/5), (1/5, 4/5), (4/5, 4/5)$, and $(4/5, 1/5)$. The outer square has vertices $(0, 0), (0, 1), (1, 1)$, and $(1, 0)$. Since the point $P$ is picked uniformly, the probability in question is just the area of the shaded region, which is

$$\left(\frac{3}{5}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{56}{225}.$$ 

If $f : [-7, 5] \to \mathbb{R}$ is the function whose graph is shown below, how many solutions does the equation $f(f(x)) = 6$ have?

Date: November 24, 2019.
**Solution:** Letting \( y = f(x) \), we see that \( f(f(x)) = 6 \) if and only if \( f(y) = 6 \), which according to the graph occurs exactly when \( y \in \{-2, 1\} \). Hence \( f(f(x)) = 6 \) if and only if \( f(x) \in \{-2, 1\} \). Drawing horizontal lines on the graph at \( y = -2 \) and \( y = 1 \), we see that \( f(x) = -2 \) for two values of \( x \) and \( f(x) = 1 \) for four values of \( x \). Naturally since \(-2 \neq 1\), these \( x \)-values must be distinct from each other, so in total we have 6 solutions to \( f(f(x)) = 6 \).

**Problem 3** (2006 AIME I Problem 3 ©MAA).
Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is \( \frac{1}{29} \) of the original integer.

**Solution:** Suppose the original number is \( N = a_na_{n-1}...a_1a_0 \), where the \( a_i \) are digits and the first digit, \( a_n \), is nonzero. Then the number we create is \( N_0 = a_{n-1}...a_1a_0 \), so
\[
N = 29N_0.
\]
But \( N \) is \( N_0 \) with the digit \( a_n \) added to the left, so \( N = N_0 + a_n \cdot 10^n \). Thus,
\[
N_0 + a_n \cdot 10^n = 29N_0
\]
\[
a_n \cdot 10^n = 28N_0.
\]
The right-hand side of this equation is divisible by seven, so the left-hand side must also be divisible by seven. The number \( 10^n \) is never divisible by 7, so \( a_n \) must be divisible by 7. But \( a_n \) is a nonzero digit, so the only possibility is \( a_n = 7 \). This gives
\[
7 \cdot 10^n = 28N_0
\]
or
\[
10^n = 4N_0.
\]
Now, we want to minimize both \( n \) and \( N_0 \), so we take \( N_0 = 25 \) and \( n = 2 \). Then
\[
N = 7 \cdot 10^2 + 25 = 725,
\]
and indeed, \( 725 = 29 \cdot 25 \).

**Problem 4** (1988 AIME ©MAA).
Suppose there is a function \( f \) defined on the set of ordered pairs \((x,y)\) of positive integers which satisfies
\[
f(x,x) = x, \quad (1)
\]
\[
f(x,y) = f(y,x), \quad \text{and} \quad (2)
\]
\[
(x+y)f(x,y) = yf(x,x+y). \quad (3)
\]
Show that there is only one possible value of \( f(14,52) \) and find it.

**Solution:** The point here is that we can use the given identities to express \( f(14,52) \) in terms of something of the form \( f(x,x) \), whose value we are given is \( x \). Since our goal is to make the arguments of the function equal, it makes sense to try to do so by making them both small; after all, there are only \( n \) positive integers less than or equal to \( n \), so our chances of finding overlap are good for small \( n \). To do so, first note that the third given equation allows us to replace one argument by the difference between the two (provided it remains positive) since
\[
(x+y-x)f(x,y-x) = (y-x)f(x,y-x+x) \implies f(x,y) = \frac{y}{y-x} f(x,y-x). \quad (4)
\]
Using this, together with the second given equation,

\[
f(14, 52) = \frac{52}{38} f(14, 38) = \frac{52}{38} \cdot \frac{38}{24} f(14, 24) = \frac{52}{38} \cdot \frac{38}{24} \cdot \frac{24}{10} f(14, 10) \]

\[
= \frac{52}{10} f(10, 14) = \frac{52}{10} \cdot \frac{14}{4} f(10, 4) \]

\[
= \frac{13 \cdot 7}{5} f(4, 10) = \frac{13 \cdot 7}{5} \cdot \frac{10}{6} f(4, 6) = \frac{13 \cdot 7}{5} \cdot \frac{10}{6} \cdot \frac{6}{2} f(4, 2) \]

\[
= 13 \cdot 7 \cdot f(2, 4) = 13 \cdot 7 \cdot \frac{4}{2} f(2, 2) \]

\[
= 13 \cdot 7 \cdot 2 \cdot 2 = 364.
\]

We remark that there is such a function, namely \( f(x, y) = \text{lcm}(x, y) \); and indeed, 364 is the least common multiple of 14 and 52.

**Problem 5.** Consider a circle with diameter \( AB \). Let \( C \) be a point outside of the circle. Suppose \( AC \) and \( BC \) intersect the circle at points \( D \) and \( M \) respectively. The areas of triangle \( DCM \) is 1/4 of the area of triangle \( ACB \). Find angle \( CBD \).

**Hint:** Triangles \( BCA \) and \( DCM \) are similar. The ratio of sides is 2. \( BC=2CD \). Since \( \sin(\text{angle CBD})=1/2 \) and the angle is acute, it equals to 30 degrees.

**Problem 6** (2008 AIME I Problem 11 @MAA).

Consider sequences that consist entirely of \( A \)'s and \( B \)'s and that have the property that every run of consecutive \( A \)'s has even length, and every run of consecutive \( B \)'s has odd length. Examples of such sequences are \( AA, B, \) and \( AABAA \), while \( BBAB \) is not such a sequence. How many such sequences have length 14?

**Solution:** (AoPS) Let \( a_n \) and \( b_n \) denote, respectively, the number of sequences of length \( n \) ending in \( A \) and \( B \). If a sequence ends in an \( A \), then it must have been formed by appending two \( A \)s to the end of a string of length \( n - 2 \). If a sequence ends in a \( B \), it must have either been formed by appending one \( B \) to a string of length \( n - 1 \) ending in an \( A \), or by appending two \( B \)s to a string of length \( n - 2 \) ending in a \( B \). Thus, we have the recursions

\[
a_n = a_{n-2} + b_{n-2} \]

\[
b_n = a_{n-1} + b_{n-2}
\]

By counting, we find that \( a_1 = 0, b_1 = 1, a_2 = 1, b_2 = 0 \).

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Therefore, the number of such strings of length 14 is \( a_{14} + b_{14} = 172 \).

**Problem 7** (2004 Manhattan Mathematical Olympiad).

Seven line segments, with lengths no greater than 10 inches, and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle.
**Problem 8** (2003 AIME I Problem 11 ©MAA).
Let $0 \leq x \leq 90$ be chosen uniformly at random. What is the probability that the numbers \( \sin^2 x, \cos^2 x, \) and \( \sin x \cos x \) do not form a triangle?

**Note:** \( x \) is measured in degrees, and you may leave your answer in terms of the \( \arctan() \) (inverse tangent) function.

**Solution:** (AoPS) First notice that the problem is symmetric about \( x = 45^\circ \) – if we replace \( x \) by \( 90 - x \), then \( \sin x \) becomes \( \cos x \) and vice versa, so we are still dealing with the same three numbers \( \sin^2 x, \cos^2 x, \) and \( \sin x \cos x \). Thus we may assume without loss of generality that \( 0 \leq x \leq 45 \). In this case, \( \cos^2 x \geq \sin x \cos x \geq \sin^2 x \), so the triangle inequality says that a triangle cannot be formed if and only if \( \sin^2 x + \sin x \cos x \leq \cos^2 x \).

Rewrite this as

\[
\frac{1}{2} \sin(2x) = \sin x \cos x \leq \sin^2 x - \cos^2 x = \cos(2x).
\]

Since \( 0 \leq x \leq 45 \), both \( \sin(2x) \) and \( \cos(2x) \) are \( \geq 0 \), so this happens if and only if

\[
\tan(2x) \leq 2 \iff x \leq \frac{1}{2} \arctan(2).
\]

Since \( x \) is picked uniformly between \( 0 \) and \( 45 \), the probability of \( x \leq \frac{1}{2} \arctan(2) \) is

\[
\frac{1}{45} \cdot \frac{\arctan(2)}{2} \approx 0.70.
\]

**Problem 9** (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom).
Prove that

\[
\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta
\]

\[
\sin 5\theta = \sin^5 \theta - 10 \sin^3 \theta \cos^2 \theta + 5 \sin \theta \cos^4 \theta
\]

Generalize those facts to

\[
\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta - \ldots
\]

\[
\sin n\theta = \binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \binom{n}{5} \cos^{n-5} \theta \sin^5 \theta - \ldots
\]
Hint: It may be helpful to recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

**Solution:** We start by expressing $\cos n\theta$ and $\sin n\theta$ as the real and imaginary parts of $e^{in\theta} = (e^{i\theta})^n$. Then expressing $e^{i\theta} = \cos \theta + i\sin \theta$, we expand with the binomial theorem, and are now looking for the real and imaginary parts of $\sum_{k=0}^{n} i^k(n) \cos^{n-k} \theta \sin^k \theta$. The real terms are those where $k$ is even, and the imaginary terms are those where $k$ is odd, so replacing $k$ with $k/2$, we get $\cos n\theta = \sum_{k=0}^{(n-1)/2} (-1)^k(n) \cos^{n-2k} \theta \sin^{2k} \theta$, and replacing $k$ with $(k-1)/2$, we get $\sin n\theta = \sum_{k=0}^{(n-1)/2} (-1)^k(n) \cos^{n-2k-1} \theta \sin^{2k+1} \theta$.

**Problem 10** (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom).
Using the previous problem, evaluate the following expressions:

\[
\cot^2 \frac{\pi}{2n+1} + \cot^2 \frac{2\pi}{2n+1} + \cdots + \cot^2 \frac{n\pi}{2n+1}
\]

\[
\csc^2 \frac{\pi}{2n+1} + \csc^2 \frac{2\pi}{2n+1} + \cdots + \csc^2 \frac{n\pi}{2n+1}
\]

**Hint:** Using the previous problem’s answer, find a polynomial with solutions $\cot \frac{k\pi}{2n+1}$ for $k = 1, \ldots, n$. Then recall that you can easily determine the sum of the roots of a polynomial from its coefficients.

**Solution:** Start by replacing $n$ in the previous problem with $2n+1$, getting $\sin ((2n+1)\theta) = \sum_{k=0}^{n} (-1)^k(n) \cos^{2n-2k} \theta \sin^{2k+1} \theta$. Then divide both sides by $\sin^{2n+1} \theta$, getting

\[
\frac{\sin((2n+1)\theta)}{\sin^{2n+1} \theta} = \sum_{k=0}^{n} (-1)^k(n) \frac{2n+1}{2k+1} \cot^{2(n-k)} \theta.
\]

Then we see that if we let $\theta = \frac{k\pi}{2n+1}$ for $1 \leq k \leq n$, the left side is 0, so $\cot^2 \theta$ is a solution of the polynomial equation $\sum_{k=0}^{n} (-1)^k(n) \frac{2n+1}{2k+1} x^{n-k}$. As this is a degree $n$ polynomial, and we have found $n$ distinct solutions, it has no more.

Now we recall that the sum of the roots of the degree $n$ polynomial $p(x) = \sum_{k=0}^{n} a_k x^n$ is $-\frac{a_{n-1}}{a_n}$, which in this case is $\frac{(2n+1)^2}{3(n+1)} = \frac{n(2n-1)}{3}$, which gives us the value of the first expression.

Then we use the identity $\cot^2 \theta + 1 = \csc^2 \theta$, showing us that the second sum is just $n$ greater, as it has $n$ terms, with value $\frac{2n(n+1)}{3}$.

**Problem 11** (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom).
Using the last problem, show that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ lies between

\[
\left(1 - \frac{1}{2n+1}\right) \left(1 - \frac{2}{2n+1}\right) \frac{\pi^2}{6} \quad \text{and} \quad \left(1 - \frac{1}{2n+1}\right) \left(1 + \frac{1}{2n+1}\right) \frac{\pi^2}{6}
\]

**Hint:** Use (and prove) the fact that if $0 < \theta < \frac{\pi}{2}$, then $\sin \theta < \theta < \tan \theta$.

**Solution:** The inequalities in the hint are best proved by drawing a picture.
We can invert the inequalities to get $\csc \theta < \frac{1}{\theta} < \cot \theta$.

**Problem 12** (2006 AIME I Problem 13 ©MAA).
For each even positive integer $x$, let $g(x)$ denote the greatest power of 2 that divides $x$. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer $n$, let $S_n = \sum_{k=1}^{2n-1} g(2k)$. Find the greatest integer $n$ less than 1000 such that $S_n$ is a perfect square.
Problem 13. Consider circumscribed circle of triangle ABC. For any point M on the circle, consider it’s projections P and Q on the sides AC and BC respectively. Find point M so that PQ is maximal.

Solution: Points M,P,Q,C lie on a circle (not the circumcircle, of course!). Thus, PQ=MC sin (angle ACB). Therefore, PQ is maximal if MC is maximal, i.e., MC is the diameter of the circumcircle going through point C.

Let $A$ be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of $n$), where $A^t$ is the transpose of $A$.

Solution: (From Kiran Kedlaya’s archive) Recall the following determinant formula. For any $d \times d$ matrix $B = (B_{i,j})$, 

$$\det(B) = \sum_{\sigma \in \text{Perm}(d)} \text{sgn}(\sigma) \prod_{i=1}^{d} B_{i,\sigma(i)}$$
where $\text{Perm}(d)$ is the set of all permutations of $\{1, \ldots, d\}$ and $\text{sgn}(\sigma) = 1$ if $\sigma$ is an even permutation and $-1$ is $\sigma$ is an odd permutation. Apply this formula to $A - A'$ and get
\[
\text{det}(A - A') = \sum_{\sigma \in \text{Perm}(2n)} \text{sgn}(\sigma) \prod_{i=1}^{2n} (A - A')_{\sigma(i)}
\]
\[
= \sum_{\sigma \in \text{Perm}(2n)} \text{sgn}(\sigma) \prod_{i=1}^{2n} (A_{\sigma(i)} - A_{\sigma(i), i}) =: \sum_{\sigma \in \text{Perm}(2n)} \text{sgn}(\sigma) \cdot E_\sigma,
\]
so by linearity of expectation
\[
\mathbb{E}(\text{det}(A - A')) = \sum_{\sigma \in \text{Perm}(2n)} \text{sgn}(\sigma) \cdot \mathbb{E}(E_\sigma).
\]
Let us examine $E_\sigma = \prod_{i=1}^{2n} (A_{i, \sigma(i)} - A_{\sigma(i), i})$. It is easiest to see what is going on by looking at an example. Suppose $n = 3$ and $\sigma = (14)(2)(356)$ in disjoint cycle notation. Then
\[
E_\sigma = (A_{1,4} - A_{4,1})(A_{2,2} - A_{2,2})(A_{3,5} - A_{5,3})(A_{4,1} - A_{1,4})(A_{5,6} - A_{6,5})(A_{6,3} - A_{3,6})
\]
\[
= (A_{1,4} - A_{4,1})(A_{4,1} - A_{1,4}) \cdot (A_{2,2} - A_{2,2}) \cdot (A_{3,5} - A_{5,3}) \cdot (A_{5,6} - A_{6,5}) \cdot (A_{6,3} - A_{3,6}).
\]
Since all of the $A_{i,j}$ are independent, the expectation splits up as
\[
\mathbb{E}(E_\sigma) = \mathbb{E}\left((A_{1,4} - A_{4,1})(A_{4,1} - A_{1,4})\right) \cdot \mathbb{E}\left((A_{2,2} - A_{2,2})\right) \cdot \mathbb{E}\left((A_{3,5} - A_{5,3})(A_{5,6} - A_{6,5})(A_{6,3} - A_{3,6})\right)
\]
\[
= \mathbb{E}\left((A_{1,4} - A_{4,1})(A_{4,1} - A_{1,4})\right) \cdot \mathbb{E}(A_{2,2} - A_{2,2}) \cdot \mathbb{E}(A_{3,5} - A_{5,3}) \cdot \mathbb{E}(A_{5,6} - A_{6,5}) \cdot \mathbb{E}(A_{6,3} - A_{3,6}).
\]
Now since each $A_{i,j}$ has the same distribution (0 with probability 1/2 and 1 with probability 1/2), all of the expectations in the above product (except for the first one) are equal to 1/2 - 1/2 = 0, so we get 0 for the entire product, so $\mathbb{E}(E_\sigma) = 0$.

However, if $\sigma$ had the form of something like $(12)(34)(56)$ in disjoint cycle notation, then all of the terms in the expression for $\mathbb{E}(E_\sigma)$ would have the same form as the first term above, and therefore would not be equal to 0. So, returning to the general case, we see that the only elements $\sigma \in \text{Perm}(2n)$ that do not have $\mathbb{E}(E_\sigma) = 0$ are those $\sigma$ of the form $(i_1 i_2)(i_3 i_4) \cdots (i_{2n-1} i_{2n})$ in disjoint cycle notation. Let $S_{2n}$ denote the set of all $\sigma \in \text{Perm}(2n)$ of this form. Then our original expression for $\mathbb{E}(\text{det}(A - A'))$ reduces to
\[
\mathbb{E}(\text{det}(A - A')) = (-1)^n \sum_{\sigma \in S_{2n}} \mathbb{E}(E_\sigma)
\]
because each $\sigma \in S_{2n}$ has $\text{sgn}$ equal to $(-1)^n$. Now, for $\sigma = (i_1 i_2)(i_3 i_4) \cdots (i_{2n-1} i_{2n}) \in S_n$, we can calculate
\[
\mathbb{E}(E_\sigma) = \mathbb{E}\left((A_{i_1 i_2} - A_{i_2 i_1})(A_{i_2 i_1} - A_{i_1 i_2})\right) \cdots \mathbb{E}\left((A_{i_{2n-1} i_{2n}} - A_{i_{2n} i_{2n-1}})(A_{i_{2n} i_{2n-1}} - A_{i_{2n-1} i_{2n}})\right)
\]
\[
= \mathbb{E}(2A_{i_1 i_2} - A_{i_1 i_2} - A_{i_2 i_1}) \cdots \mathbb{E}(2A_{i_{2n-1} i_{2n}} - A_{i_{2n-1} i_{2n}} - A_{i_{2n} i_{2n-1}})
\]
\[
= (2(1/2)(1/2) - 1/2 - 1/2) \cdots (2(1/2)(1/2) - 1/2 - 1/2) = (1/2)^n (-1)^n.
\]
Therefore we have
\[
\mathbb{E}(\text{det}(A - A')) = (-1)^n \sum_{\sigma \in S_{2n}} \mathbb{E}(E_\sigma) = (-1)^n \sum_{\sigma \in S_{2n}} 2^{-n}(-1)^n = \frac{|S_{2n}|}{2^n}.
\]
Now we must just count the number of $\sigma \in \text{Perm}(2n)$ of the form $(i_1 i_2)(i_3 i_4) \cdots (i_{2n-1} i_{2n})$. This number is simply equal to the number of ways to split a group of $2n$ objects into $n$ groups of 2, which is $\frac{(2n)!}{n!2^n}$, and therefore the final answer is
\[
\mathbb{E}(\text{det}(A - A')) = \frac{(2n)!}{n!4^n}.
\]