Problem 1 (2010 AIME II Problem 2 ©MAA).
A point \( P \) is chosen at random in the interior of a unit square \( S \). Let \( d(P) \) denote the distance from \( P \) to the closest side of \( S \). Find the probability that \( 1/5 \leq d(P) \leq 1/3 \).

If \( f : [-7, 5] \to \mathbb{R} \) is the function whose graph is shown below, how many solutions does the equation \( f(f(x)) = 6 \) have?

![Graph of f(x) showing points (-2, 6), (1, 6), (-7, -4), (5, -6)]

Problem 3 (2006 AIME I Problem 3 ©MAA).
Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is \( \frac{1}{29} \) of the original integer.

Suppose there is a function \( f \) defined on the set of ordered pairs \((x, y)\) of positive integers which satisfies

\[
\begin{align*}
    f(x, x) &= x, \\
    f(x, y) &= f(y, x), \quad \text{and} \\
    (x + y) f(x, y) &= y f(x, x + y).
\end{align*}
\]

Show that there is only one possible value of \( f(14, 52) \) and find it.

Problem 5. Consider a circle with diameter \( AB \). Let \( C \) be a point outside of the circle. Suppose \( AC \) and \( BC \) intersect the circle at points \( D \) and \( M \) respectively. The areas of triangle \( DCM \) is \( 1/4 \) of the area of triangle \( ACB \). Find angle \( CBD \).

Problem 6 (2008 AIME I Problem 11 ©MAA).
Consider sequences that consist entirely of \( A \)'s and \( B \)'s and that have the property that every run of consecutive \( A \)'s has even length, and every run of consecutive \( B \)'s has odd length. Examples of such sequences are \( AA, B, \) and \( AABAA \), while \( BBAB \) is not such a sequence. How many such sequences have length 14?

Date: November 24, 2019.
Problem 7 (2004 Manhattan Mathematical Olympiad).
Seven line segments, with lengths no greater than 10 inches, and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle.

Problem 8 (2003 AIME I Problem 11 ©MAA).
Let $0 \leq x \leq 90$ be chosen uniformly at random. What is the probability that the numbers $\sin^2 x$, $\cos^2 x$, and $\sin x \cos x$ do not form a triangle?

**Note:** $x$ is measured in degrees, and you may leave your answer in terms of the arctan() (inverse tangent) function.

Problem 9 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom).
Prove that
\[
\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta
\]
\[
\sin 5\theta = \sin^5 \theta - 10 \sin^3 \theta \cos^2 \theta + 5 \sin \theta \cos^4 \theta
\]

Generalize those facts to
\[
\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta - \ldots
\]
\[
\sin n\theta = \binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \binom{n}{5} \cos^{n-5} \theta \sin^5 \theta - \ldots
\]

**Hint:** It may be helpful to recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Problem 10 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom).
Using the previous problem, evaluate the following expressions:
\[
\cot^2 \frac{\pi}{2n+1} + \cot^2 \frac{2\pi}{2n+1} + \cdots + \cot^2 \frac{n\pi}{2n+1}
\]
\[
\csc^2 \frac{\pi}{2n+1} + \csc^2 \frac{2\pi}{2n+1} + \cdots + \csc^2 \frac{n\pi}{2n+1}
\]

**Hint:** Using the previous problem’s answer, find a polynomial with solutions $\cot^2 \frac{k\pi}{2n+1}$ for $k = 1, \ldots, n$. Then recall that you can easily determine the sum of the roots of a polynomial from its coefficients.

Problem 11 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom).
Using the last problem, show that $1 + \frac{1}{\pi^2} + \frac{1}{2^2 \pi^2} + \cdots + \frac{1}{n^2 \pi^2}$ lies between
\[
\left(1 - \frac{1}{2n+1}\right) \left(1 - \frac{2}{2n+1}\right) \frac{\pi^2}{6} \quad \text{and} \quad \left(1 - \frac{1}{2n+1}\right) \left(1 + \frac{1}{2n+1}\right) \frac{\pi^2}{6}
\]

**Hint:** Use (and prove) the fact that if $0 < \theta < \frac{\pi}{2}$, then $\sin \theta < \theta < \tan \theta$.

For each even positive integer $x$, let $g(x)$ denote the greatest power of 2 that divides $x$. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer $n$, let $S_n = \sum_{k=1}^{2n-1} g(2k)$. Find the greatest integer $n$ less than 1000 such that $S_n$ is a perfect square.

Problem 13. Consider circumscribed circle of triangle ABC. For any point M on the circle, consider it’s projections P and Q on the sides AC and BC respectively. Find point M so that the length of PQ is as big as possible.

Let $A$ be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A^t)$ (as a function of $n$), where $A^t$ is the transpose of $A$. 