1. Let \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \) be a complex polynomial with real coefficients. This means that the function takes in values \( z \in \mathbb{C} \) but the coefficients \( a_i \) are real. For example, for any real polynomial we can just extend the function to complex inputs.

(a) Show that the complex conjugate of \( f(z) \), \( \overline{f(z)} \), is equal to \( f(\overline{z}) \). (Hint: Use the multiplicative property of the conjugate).

(b) Suppose \( z_0 \) is a root of \( f(z) \). This means that \( f(z_0) = 0 \). Use part (a) to show that \( \overline{z}_0 \) is also a root of \( f(z) \).

(c) What does your result in part (b) tell you about complex roots to polynomial equations?
2. Recall the polar form of a complex number: \(re^{i\theta}\), where \(r \in \mathbb{R}\), \(\theta \in [0, 2\pi)\).

(a) Use polar form to find all complex solutions to the equation \(z^3 = 1\). Write your solutions in the form \(a + bi\), where \(a, b \in \mathbb{R}\).

(b) Find all complex solutions to the equation \(z^5 = 2\). You can leave your answers in polar form.

(c) Let \(n \in \mathbb{N}\). Find all complex solutions to the equation \(z^n = 1\). You can leave your answers in polar form. Find a root \(\omega_n \in \mathbb{C}\) such that the set \(\{\omega_n, \omega_n^2, ..., \omega_n^n\}\) are all of the solutions to \(z^n = 1\).

(d) Draw the roots of \(z^4 - 1\) in the complex plane. If you connect adjacent roots with lines, what shape does this form? What shape would the roots to \(z^n = 1\) make?
Definition 1.
For any positive integer \( n \), the \( n^{th} \) roots of unity are the complex solutions to the equation \( z^n = 1 \).

3. (a) Show that if \( \omega \in \mathbb{C} \) is an \( n^{th} \) root of unity, then so is \( \bar{\omega} \).

(b) Let \( \omega_5 = e^{\frac{2\pi i}{5}} \). Show that \( \{\omega_5, \omega_5^2, \ldots, \omega_5^5\} \) are all solutions to \( z^5 = 1 \). Are there other solutions?

(c) Use part (b) to factor \( f(z) = z^5 - 1 \).

(d) By expanding out your factored form and comparing with the expanded version, show that \( \sum_{i=1}^{5} \omega_5^i = 0 \).

(e) Generalize parts (a) - (c) to show that the sum of the roots of \( z^n - 1 \) is equal to 0.
4. Again consider $\omega_5 = e^{\frac{2\pi i}{5}}$.

(a) Use the formula for $e^{i\theta}$ from last time to show that $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$.

(b) Show that $\omega_i^5 = \omega_5^{5-i}$ for $i \in \{1, 2, 3, 4, 5\}$.

(c) Let $x = \omega_5 + \bar{\omega}_5$. Compute and simplify $x^2 + x$. (Hint: Use your result from problem 3(d)).

(d) Find the exact value of $\cos\left(\frac{2\pi}{5}\right) = \cos(72^\circ)$. Hint: Use parts (a) and (b).
5. Suppose we consider the n-1 diagonals of a regular n-gon inscribed in a unit circle by connecting one vertex with all the others (an edge connecting adjacent vertices is considered a diagonal here). Show that the product of their lengths is n by following the steps below:

(a) Draw the n-gon in the complex plane, where the vertex connecting to all the others is at 1. Show that all the other vertices \( \{z_1, ..., z_{n-1}\} \) are the rest of the \( n^{th} \) roots of unity. Write an equation for the product of the lengths of the diagonals in terms of the \( z_i \).

(b) Consider the equation \( z^n - 1 = 0 \). Factor out the term corresponding to the solution \( z = 1 \) and show that all other \( n^{th} \) roots of unity \( z_i \) satisfy \( z_i^{n-1} + z_i^{n-2} + ... + z_i + 1 = 0 \). Conclude that the \( z_i \) are the roots of the equation \( z^{n-1} + z^{n-2} + ... + z + 1 \).

(c) Shifting the n-gon to the left one unit, represent the new vertices \( w_i \) in terms of the old ones \( z_i \). What equation represents the product of the lengths of the diagonals now?

(d) Change the equation in part (b) after shifting the n-gon to the left by one so that the new vertices \( w_i \) are now the solutions. Use a fact relating the product of the roots of a polynomial to the constant term of the polynomial to show that the product of the lengths of the diagonals is n.
6. Consider a regular n-gon inscribed in a unit circle. Connect every pair of vertices with a diagonal. Find the product of the lengths of all diagonals. (Again, an edge connecting adjacent vertices is considered a diagonal here.)

7. Prove that if the consecutive vertices \( z_1, z_2, z_3, z_4 \) of any quadrilateral lie on a circle, then \( |z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_1 - z_4||z_2 - z_3| \). Don’t just state Ptolemy’s Theorem. Hints:

(a) Show that \((z_1 - z_3)(z_2 - z_4) = (z_1 - z_2)(z_3 - z_4) + (z_1 - z_4)(z_2 - z_3)\) for complex numbers \( z_1, z_2, z_3, z_4 \).

(b) Let \( z, w \in \mathbb{C} \). When does \(|z + w| = |z| + |w|\)?