Complex Numbers II

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1. Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ be a complex polynomial with real coefficients. This means that the function takes in values $z \in \mathbb{C}$ but the coefficients a_i are real. For example, for any real polynomial we can just extend the function to complex inputs.

   (a) Show that the complex conjugate of $f(z)$, $\overline{f(z)}$, is equal to $f(\bar{z})$. (Hint: Use the multiplicative property of the conjugate).

   (b) Suppose z_0 is a root of $f(z)$. This means that $f(z_0) = 0$. Use part (a) to show that $\bar{z}_0$ is also a root of f(z).

   (c) What does your result in part (b) tell you about complex roots to polynomial equations?

2. Recall the polar form of a complex number: $re^{i\theta}$, where $r \in \mathbb{R}, \theta \in [0, 2\pi)$.

(a) Use polar form to find all complex solutions to the equation $z^3 = 1$. Write your solutions in the form $a + bi$, where $a, b \in \mathbb{R}$.

(b) Find all complex solutions to the equation $z^5 = 2$. You can leave your answers in polar form.

(c) Let $n \in \mathbb{N}$. Find all complex solutions to the equation $z^n = 1$. You can leave your answers in polar form. Find a root $\omega_n \in \mathbb{C}$ such that the set $\{\omega_n, \omega_n^2, ..., \omega_n^n\}$ are all of the solutions to $z^n = 1$.

(d) Draw the roots of $z^4 - 1$ in the complex plane. If you connect adjacent roots with lines, what shape does this form? What shape would the roots to $z^n = 1$ make?

Definition 1.
For any positive integer n, the n^{th} roots of unity are the complex solutions to the equation $z^n = 1$.

3. (a) Show that if $\omega \in \mathbb{C}$ is an n^{th} root of unity, then so is $\bar{\omega}$.

   (b) Let $\omega_5 = e^{\frac{2\pi i}{5}}$. Show that $\{\omega_5, \omega_5^2, ..., \omega_5^5\}$ are all solutions to $z^5 = 1$. Are there other solutions?

   (c) Use part (b) to factor $f(z) = z^5 - 1$.

   (d) By expanding out your factored form and comparing with the expanded version, show that $\sum_{i=1}^{5} \omega_5^i = 0$.

   (e) Generalize parts (a) - (c) to show that the sum of the roots of $z^n - 1$ is equal to 0.

4. Again consider $\omega_5 = e^{\frac{2\pi i}{5}}$.

(a) Use the formula for $e^{i\theta}$ from last time to show that $cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

(b) Show that $\overline{\omega_5^i} = \omega_5^{5-i}$ for $i \in \{1, 2, 3, 4, 5\}$.

(c) Let $x = \omega_5 + \bar{\omega}_5$. Compute and simplify $x^2 + x$. (Hint: Use your result from problem 3(d)).

(d) Find the exact value of $cos(\frac{2\pi}{5}) = cos(72°)$. Hint: Use parts (a) and (b).

5. Suppose we consider the n-1 diagonals of a regular n-gon inscribed in a unit circle by connecting one vertex with all the others (an edge connecting adjacent vertices is considered a diagonal here). Show that the product of their lengths is n by following the steps below:

   (a) Draw the n-gon in the complex plane, where the vertex connecting to all the others is at 1. Show that all the other vertices $\{z_1, ..., z_{n-1}\}$ are the rest of the n^{th} roots of unity. Write an equation for the product of the lengths of the diagonals in terms of the z_i.

   (b) Consider the equation $z^n - 1 = 0$. Factor out the term corresponding to the solution $z = 1$ and show that all other n^{th} roots of unity z_i satisfy $z_i^{n-1} + z_i^{n-2} + ... + z_i + 1 = 0$. Conclude that the z_i are the roots of the equation $z^{n-1} + z^{n-2} + ... + z + 1$.

   (c) Shifting the n-gon to the left one unit, represent the new vertices w_i in terms of the old ones z_i. What equation represents the product of the lengths of the diagonals now?

   (d) Change the equation in part (b) after shifting the n-gon to the left by one so that the new vertices w_i are now the solutions. Use a fact relating the product of the roots of a polynomial to the constant term of the polynomial to show that the product of the lengths of the diagonals is n.

6. Consider a regular n-gon inscribed in a unit circle. Connect every pair of vertices with a diagonal. Find the product of the lengths of all diagonals. (Again, an edge connecting adjacent vertices is considered a diagonal here.)

7. Prove that if the consecutive vertices z_1, z_2, z_3, z_4 of **any** quadrilateral lie on a circle, then $|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_1 - z_4||z_2 - z_3|$. Don't just state Ptolemy's Theorem. Hints:

   (a) Show that $(z_1 - z_3)(z_2 - z_4) = (z_1 - z_2)(z_3 - z_4) + (z_1 - z_4)(z_2 - z_3)$ for complex numbers z_1, z_2, z_3, z_4.

   (b) Let $z, w \in \mathbb{C}$. When does $|z + w| = |z| + |w|$?.