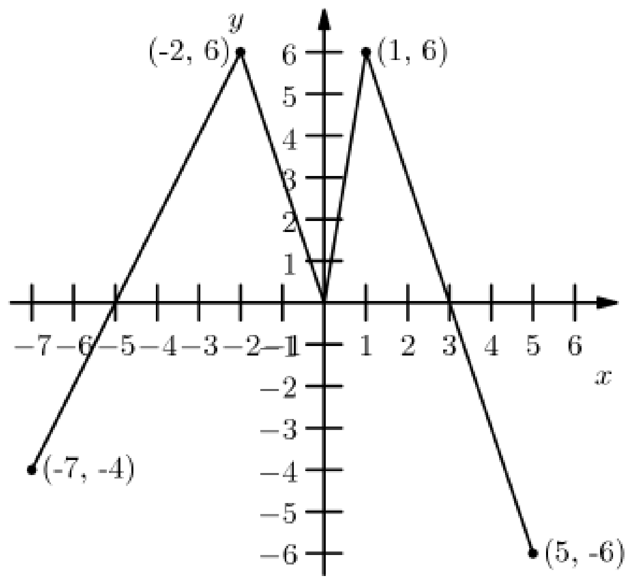
OLYMPIAD-STYLE PROBLEMS II

COLLECTED FOR THE
LOS ANGELES MATH CIRCLE

Problem 1 (2010 AIME II Problem 2 ©MAA)**.**
A point P is chosen at random in the interior of a unit square S. Let $d(P)$ denote the distance from P to the closest side of S. Find the probability that $1/5 \le d(P) \le 1/3$.

Problem 2 (2002 AMC 12A ©MAA)**.**
If $f : [-7, 5] \to \mathbb{R}$ is the function whose graph is shown below, how many solutions does the equation $f(f(x)) = 6$ have?



Problem 3 (2006 AIME I Problem 3 ©MAA)**.**
Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

Problem 4 (1988 AIME ©MAA)**.**
Suppose there is a function f defined on the set of ordered pairs (x, y) of positive integers which satisfies

$$f(x, x) = x, \tag{1}$$
$$f(x, y) = f(y, x), \quad \text{and} \tag{2}$$
$$(x + y)f(x, y) = yf(x, x + y). \tag{3}$$

Show that there is only one possible value of $f(14, 52)$ and find it.

Problem 5. Consider a circle with diameter AB. Let C be a point outside of the circle. Suppose AC and BC intersect the circle at points D and M respectively. The areas of triangle DCM is 1/4 of the area of triangle ACB. Find angle CBD.

Problem 6 (2008 AIME I Problem 11 ©MAA)**.**
Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?

Date: November 24, 2019.

Problem 7 (2004 Manhattan Mathematical Olympiad)**.**
Seven line segments, with lengths no greater than 10 inches, and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle.

Problem 8 (2003 AIME I Problem 11 ©MAA)**.**
Let $0 \leq x \leq 90$ be chosen uniformly at random. What is the probability that the numbers $\sin^2 x$, $\cos^2 x$, and $\sin x \cos x$ do not form a triangle?

Note: x is measured in degrees, and you may leave your answer in terms of the arctan() (inverse tangent) function.

Problem 9 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.**
Prove that

$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$$
$$\sin 5\theta = \sin^5 \theta - 10 \sin^3 \theta \cos^2 \theta + 5 \sin \theta \cos^4 \theta$$

Generalize those facts to

$$\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta - \dots$$
$$\sin n\theta = \binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \binom{n}{5} \cos^{n-5} \theta \sin^5 \theta - \dots$$

Hint: It may be helpful to recall that $e^{i\theta} = \cos(\theta) + i \sin(\theta)$.

Problem 10 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.**
Using the previous problem, evaluate the following expressions:

$$\cot^2 \frac{\pi}{2n+1} + \cot^2 \frac{2\pi}{2n+1} + \dots + \cot^2 \frac{n\pi}{2n+1}$$
$$\csc^2 \frac{\pi}{2n+1} + \csc^2 \frac{2\pi}{2n+1} + \dots + \csc^2 \frac{n\pi}{2n+1}$$

Hint: Using the previous problem's answer, find a polynomial with solutions $\cot^2 \frac{k\pi}{2n+1}$ for $k = 1, \dots, n$. Then recall that you can easily determine the sum of the roots of a polynomial from its coefficients.

Problem 11 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.**
Using the last problem, show that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$ lies between

$$\left(1 - \frac{1}{2n+1}\right)\left(1 - \frac{2}{2n+1}\right)\frac{\pi^2}{6} \quad \text{and} \quad \left(1 - \frac{1}{2n+1}\right)\left(1 + \frac{1}{2n+1}\right)\frac{\pi^2}{6}$$

Hint: Use (and prove) the fact that if $0 < \theta < \frac{\pi}{2}$, then $\sin \theta < \theta < \tan \theta$.

Problem 12 (2006 AIME I Problem 13 ©MAA)**.**
For each even positive integer x, let $g(x)$ denote the greatest power of 2 that divides x. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer n, let $S_n = \sum_{k=1}^{2^{n-1}} g(2k)$. Find the greatest integer n less than 1000 such that S_n is a perfect square.

Problem 13. Consider circumscribed circle of triangle ABC. For any point M on the circle, consider it's projections P and Q on the sides AC and BC respectively. Find point M so that the length of PQ is as big as possible.

Problem 14 (2016 Putnam Problem B4 ©MAA)**.**
Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.