Gnomons (Part II):
Triangular and Square Numbers

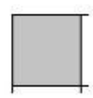
May 17, 2015

1. Triangular Numbers

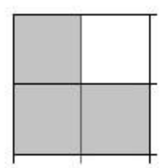
1. Take a look at the "triangular" made out of blocks (see pictures below). Notice that the "height" and the "width" of a triangle are the same.

   The number of squares making each of the "triangles" is called a triangular number.

   (a) The first triangular number, which we denote by T_1, is 1:

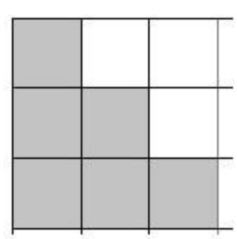
   

   (b) The second triangular number, which we denote by T_2, is 3:

   (c) Explain why 2 is not a triangular number (Hint: can we make a triangle out of just two squares?) You will not be able to make a triangle out of just two squares.

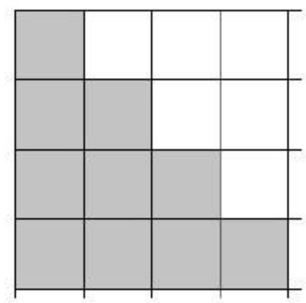
   (d) The third triangular number, which we write as T_3, is 6.

(e) Explain why 4 and 5 are not triangular numbers.

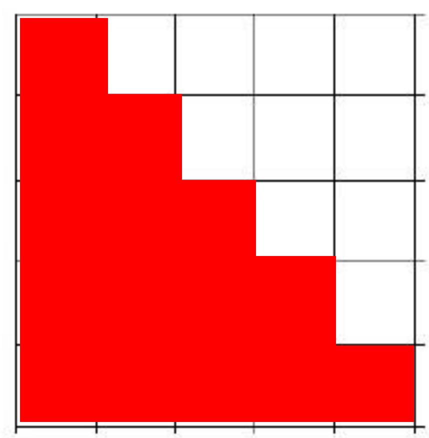
Neither can make a “triangle” in which the height and width are the same.

(f) What is T_4, the fourth triangular number? 10



(g) What is T_5, the fifth triangular number? (You may find it helpful to draw the next-largest block triangle in the triangle below.)

5+4+3+2+1 = **15**



2. Remember from last week our definitions of similar shapes: two shapes are similar when the larger is a "magnified" version of the smaller shape.

   (a) Compare the shapes corresponding to T_3 and T_4 (see previous pictures).

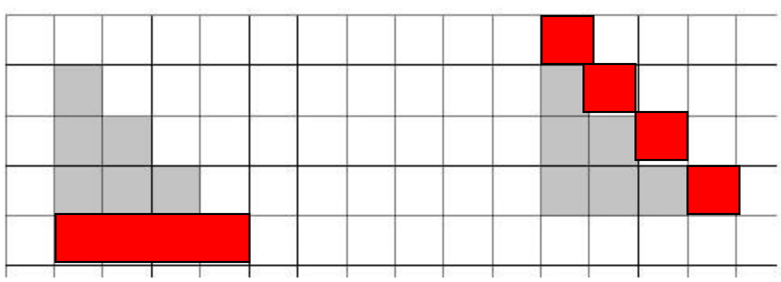
      i. What do they have in common?

         Both look like "triangles"

      ii. Do you think that they are similar shapes by our definition from last week? (You need to check if T_4 can be obtained by magnifying T_3).

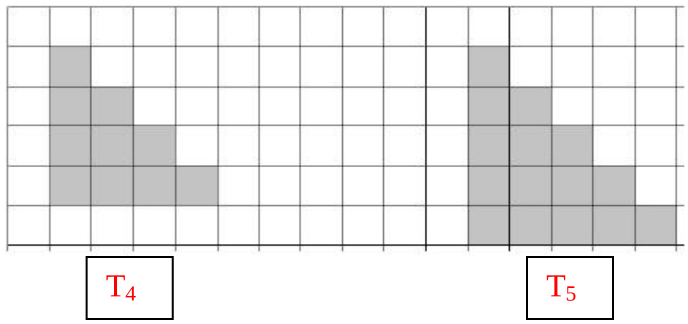
         From last week, our definition was that two shapes are similar if one is a magnified version of the other in both length and width. T_4 cannot be obtained by magnifying T_3, so they are not similar.

   (b) Even though the T_3-triangle and the T_4-triangle are not similar, we can speak of a gnomon that transforms T_3-triangle into T_4-triangle. Draw two different gnomons next to each of the T_3-triangles below that make it into a T_4-triangle. (Position one of the gnomons "below" the T_3-triangle and the other along the "stairs" of the T_3-triangle).



      i. What is the size of the gnomon that you need to transform the T_3 -triangle into the T_4-triangle? 4 squares

3. For now, consider only the gnomons positioned along the “stairs”. Below are two more block triangles.

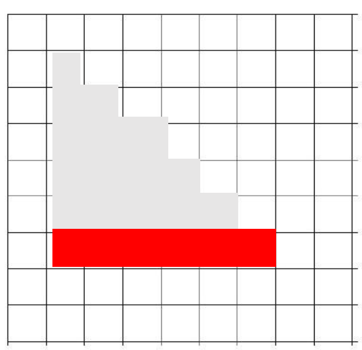


   (a) What triangular numbers do the block triangles correspond to? Label them below the triangle.

   (b) How many blocks did you add this time? (What is the size of the gnomon that transforms the T_4-triangle into the T_5-triangle?) 5 blocks

4. Can you guess the size of the gnomon that would be needed to turn the T_5-triangle into the T_6-triangle? 6 blocks

   (a) Draw it out and verify your answer.



   (b) What do you think about the pattern of the sizes of gnomons needed to change a T_n-triangle into a T_{n+1}-triangle? How is the triangular number associated with a T_n-triangle related to the size of the gnomon needed to form a T_{n+1}-triangle? Gnomon needed is (n+1).

5. Write down the first 6 triangular numbers:

   (a) T_1= **1**

   (b) T_2= **3**

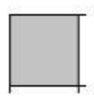
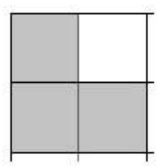
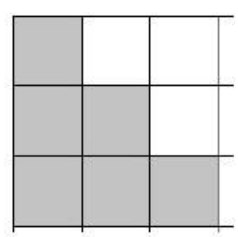
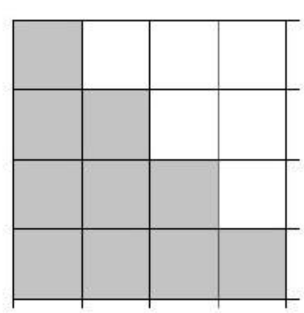
   (c) T_3= **6**

   (d) T_4= **10**

   (e) T_5= **15**

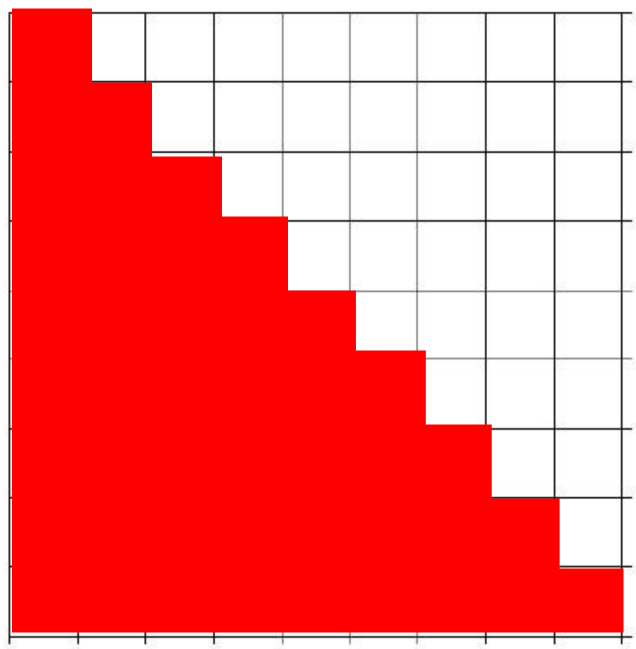
   (f) T_6=**21**

6. Represent each triangular number as the sum of the numbers of squares in the rows of the corresponding block triangle (the first two are done for you).

	Triangle	Number	Sum of the lengths of each of the rows
T_1		1	1
T_2		3	1+2
T_3		**6**	3+2+1
T_4		**10**	4+3+2+1

(a) Can you say what the 9th triangular number is without drawing a triangle? 9+8+7+6+5+4+3+2+1= **45**

7. Draw a triangle corresponding to T_9 and verify the answer above.



8. Jeremy spend the weekend drawing the triangle representing T_{100}. After counting all the blocks carefully, he discovered that

$$T_{100} = 5{,}050.$$

(a) Assuming that Jeremy is correct, can you help him figure out the value of T_{101}?

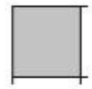
$$T_{101} = T_{100} + 101 = \mathbf{5{,}151}$$

(b) Assuming that Jeremy is correct, can you help him figure out the value of T_{99}? $T_{99} = T_{100} - 100 =$ **4,950**

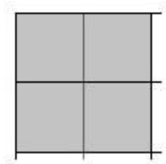
2. Square Numbers

There is another special set of numbers known as square numbers. As you might guess from their name, these numbers represent the number of blocks contained inside of a square.

1. The first square number, which we denote S_1 is 1



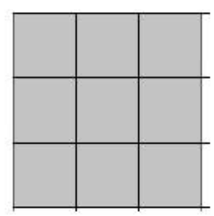
2. The second square number, which we note by S_2, is 4



3. Why are 2 and 3 not square numbers?

   2 or 3 squares cannot be put together to make a square

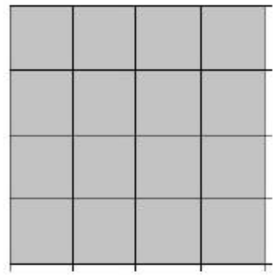
4. The third square number, which we write as S_3, is 9.



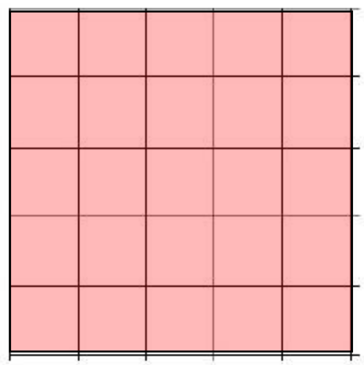
5. Why are 5, 6, 7 and 8 not square numbers?

   Similar to the numbers 2 and 3, those numbers of squares cannot be put together to make a square

6. What is S_4, the fourth square number? 16



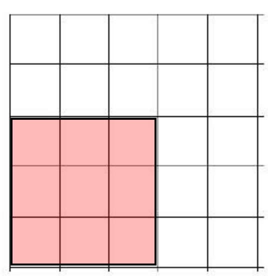
7. What is S_5, the fifth square number? (You may find it helpful to draw the next-largest block square in the square below.)



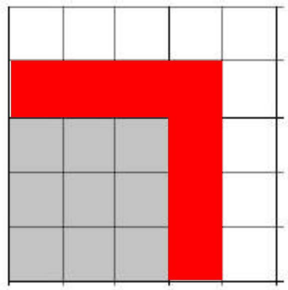
8. Remember from last week our definitions of similar shapes: two shapes are similar when the larger is a "magnified" version of the smaller shape.

   (a) Do you think all squares are similar each other by this definition? Why or why not?
   Yes, each side of the square is magnified by the other by the same magnitude (as both sides of the square are equal)

   (b) Draw the square corresponding to S_3.

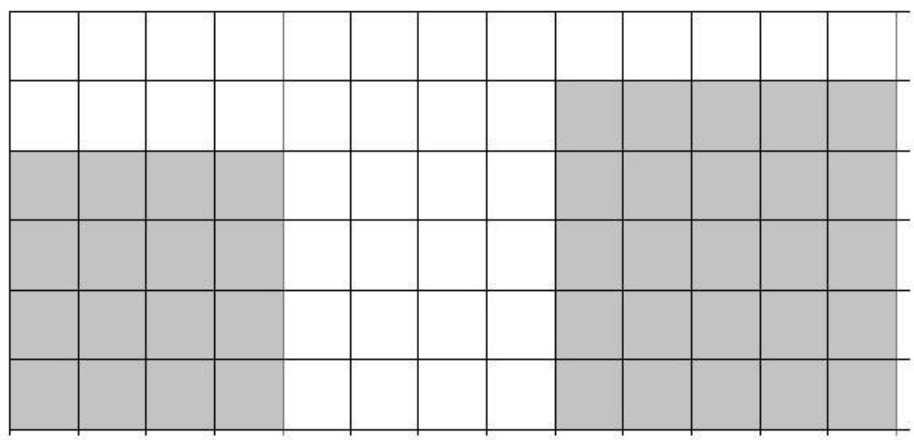


(c) Draw a gnomon that transforms S_3-square below into an S_4-square.



(d) What is the size of the gnomon? 7 squares

9. Below are two more squares.



S_4

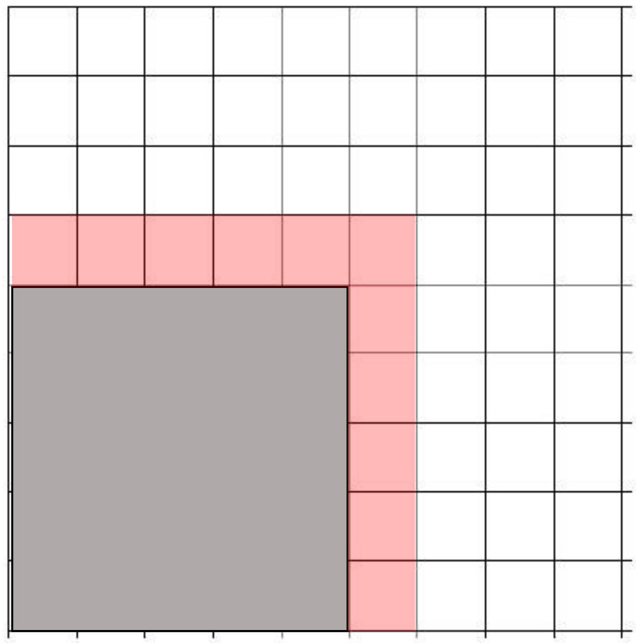
S_5

(a) What square numbers do the block squares correspond to? Label them below the squares.

(b) How many blocks did you add this time? (What is the size of the gnomon that transforms S_4-square below into an S_5-square?) 9

10. Can you guess the size of the gnomon that would be needed to turn the S_5-square into the S_6-square? 11

11. Make a picture to check your answer to the previous problem.



(a) What do you notice about the pattern of the size of gnomons when we change an S_n-square into an S_{n+1}-square. How is the square number associated with an S_n-square related to the size of the gnomon needed to make an S_{n+1}-square? n + (n+1) = **2n+1**

12. Write down the first 5 square numbers:

(a) S_1= 1

(b) S_2= 4

(c) S_3= 9

(d) S_4= 16

(e) $S_5 = 25$

13. Can you say what the 9th square number is without drawing the square?

$9^2 = \mathbf{81}$

14. Jenna computed the 50th square number, $S_{50} = 2500$.

(a) Given this, can you find S_{51}? (Hint: think about the size of the gnomon).

$$S_{51} = 2500 + (50+51) = \mathbf{2601}$$

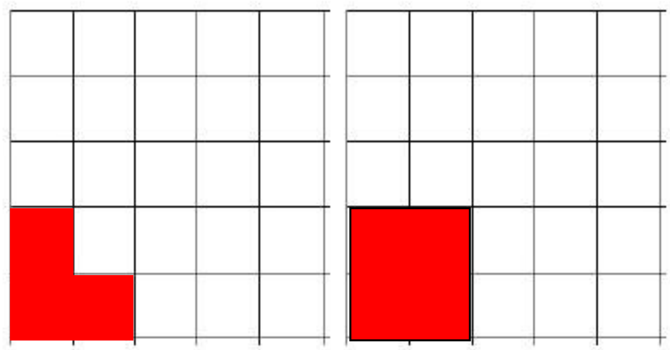
(b) Given this, can you find S_{49}?

$$S_{49} = 2500 - (49+50) = \mathbf{2401}$$

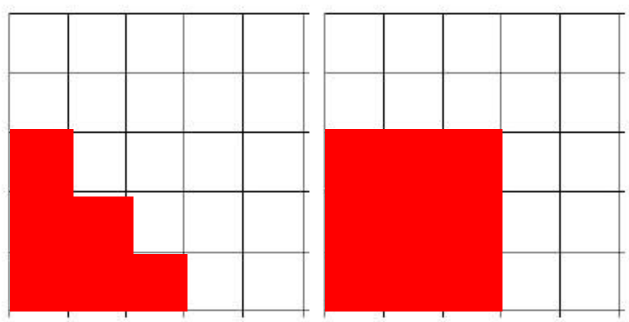
3. How are triangular and square numbers related?

1. Draw the pictures representing several triangular numbers next to the pictures representing the square numbers:

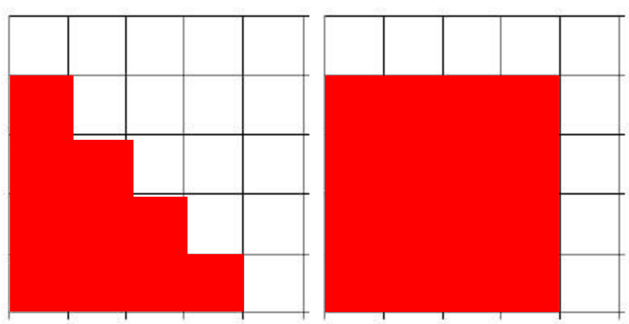
   (a) T_2 and S_2



   (b) T_3 and S_3



   (c) T_4 and S_4



2. Looking at the pictures in the previous problem, describe what shape you need to attach to a block triangle to get a square. Attach T_1, T_2, T3, respectively

3. Compute the following:

$$T_2 + T_1 = S_2$$
$$T_3 + T_2 = S_3$$
$$T_4 + T_3 = S_4$$
$$T_5 + T_4 = S_5$$

$$S_5 - T_5 = T_4$$
$$S_4 - T_3 = T4$$

4. Jeremy told Jenna that the 98th triangular number is 4,851 and the 99th triangular number is 4,950. Using these two pieces of information, Jenna was able to compute the 99th square number right away.
   (a) How did Jenna solve the problem?
   T_{98}=4851
   T_{99}=4950
   From the previous problem, recognize the pattern that T_n+T_{n-1} = S_n

   $T_{99}+T_{98}=S_{99}=4851+4950=$**9801**

   (b) Check that Jenna's answer for the 99th square number is correct. (Hint: Recall what square numbers represent geometrically).
   99^2= **9801**
   Jenna's answer is correct.

5. Find a number that is both a triangular number and a square number. 1

4. Homework

1. Find a real-life example where triangular numbers or square numbers are important. Come next week prepared to share this example with your table.