# TRIGONOMETRY IN OLYMPIAD GEOMETRY

LAMC OLYMPIAD GROUP, WEEK 8A

Trigonometry is the study of lengths in geometrical configurations, when certain angles are known. It turns out that the whole study can be reduced to finding lengths in a right triangle.

We begin by defining the sine and cosine functions for angles $0 \leqslant \theta \leqslant 90^\circ$. Given a right triangle $\triangle ABC$ with $\angle B = 90^\circ$ and $\angle A = \theta$, we make the definitions

$$\sin\theta \coloneqq \frac{BC}{AC}, \quad \cos\theta \coloneqq \frac{AB}{AC}.$$

These do not depend on the choice of the triangle, due to similarity. When fractions make sense, one also defines the tangent, cotangent, secant, cosecant:

$$\tan\theta \coloneqq \frac{\sin\theta}{\cos\theta}, \quad \cot\theta \coloneqq \frac{\cos\theta}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}.$$

The last two are seldom used. Here is a table for common values:

| | $0^\circ$ | $15^\circ$ | $18^\circ$ | $30^\circ$ | $36^\circ$ | $45^\circ$ | $54^\circ$ | $60^\circ$ | $72^\circ$ | $75^\circ$ | $90^\circ$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $\frac{\sqrt{3}-1}{2\sqrt{2}}$ | $\frac{\sqrt{5}-1}{4}$ | $\frac{1}{2}$ | $\sqrt{\frac{5-\sqrt{5}}{8}}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{5}+1}{4}$ | $\frac{\sqrt{3}}{2}$ | $\sqrt{\frac{5+\sqrt{5}}{8}}$ | $\frac{\sqrt{3}+1}{2\sqrt{2}}$ | $1$ |
| $\cos\theta$ | $1$ | $\frac{\sqrt{3}+1}{2\sqrt{2}}$ | $\sqrt{\frac{5+\sqrt{5}}{8}}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{5}+1}{4}$ | $\frac{1}{\sqrt{2}}$ | $\sqrt{\frac{5-\sqrt{5}}{8}}$ | $\frac{1}{2}$ | $\frac{\sqrt{5}-1}{4}$ | $\frac{\sqrt{3}-1}{2\sqrt{2}}$ | $0$ |

The values of sine and cosine can be extended to the whole real line using Cartesian coordinates on the unit circle. These are subject to the relations:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin^2\theta + \cos^2\theta = 1.$$

Applying the transformation $\theta \longmapsto \theta + 90^\circ$ cycles through the functions

$$\boxed{\sin} \longmapsto \cos \longmapsto -\sin \longmapsto -\cos \longmapsto \boxed{\sin} \cdots$$

while $\theta \longmapsto 90^\circ - \theta$ simply swaps sine and cosine. In particular, one also has

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos(\theta),$$
$$\boldsymbol{\sin(180^\circ - \theta) = \sin\theta}.$$

It is also useful to know that on the interval $[0^\circ, 90^\circ]$, the function sin is strictly increasing, and the function cos is strictly decreasing.

Perhaps the most important way to relate lengths to trigonometric functions is the **law of sines**: in a triangle $\triangle ABC$ with angles denoted $A, B, C$, side-lengths denoted $a, b, c$ and the radius of the circumcircle equal to $R$, one has

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

In the same setting, the **law of cosines** states that

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

We can also relate areas to trigonometric functions; in the previous setting, the area of $\triangle ABC$ is

$$[ABC] = \frac{b \cdot c \cdot \sin A}{2} = \frac{c \cdot a \cdot \sin B}{2} = \frac{a \cdot b \cdot \sin C}{2} = \frac{abc}{4R}.$$

Finally, in a circle of radius $R$, the length of a chord determined by an angle $\theta < 180°$ (at the center of the circle) is

$$2R \cdot \sin\left(\frac{\theta}{2}\right),$$

which follows from the law of sines. Compare this to the length of the arc determined by $\theta$, which is $2\pi R \frac{\theta}{360°}$.

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**Problem 1.** Let $ABCD$ be a convex quadrilateral, $X$ be the intersection of the diagonals, and $\alpha := \angle AXB$. Show that the area of $ABCD$ is $\frac{AC \cdot BD \sin\alpha}{2}$.

**Problem 2.** In a triangle $\triangle ABC$, points $D$ and $E$ lie on $BC$ (in the order $B, D, E, C$) such that $\angle BAD = \angle CAE$ and $BD = CE$. Show that $\triangle ABC$ is isosceles.

*Hint: Try showing that* $\frac{AD}{AE} = \frac{AC}{AB}$ *using the law of sines.*

**Problem 3 (Angle Bisector Theorem).** Show, using the law of sines, that in a triangle $\triangle ABC$ with bisector $AD$ ($D \in BC$), one has $\frac{AB}{BD} = \frac{AC}{CD}$.

**Problem 4.** Given an equliateral triangle $\triangle ABC$ inscribed in a circle $\mathcal{C}$ and a point $P$ on the shorter arc $BC$, show that $AP = BP + CP$. *Hint: Try expressing the three lengths in terms of cosines of angles.*

**Problem 5.** Let $AM$ be a median in an acute triangle $\triangle ABC$ ($M \in BC$, $BM = MC$) and $D$ be on the opposite side of $BC$ such that $\angle CBD = \angle BCD = \angle BAC$. Show that $\angle BAD = \angle CAM$ (in other words, the line $AD$ is the *symmedian*).

**Problem 6.** **(a)** Compute $\sin 15°$ using $\sin 30° = \frac{1}{2}$ and a quadratic equation.

**(b)** Compute $\sin 36°$ from the following construction: let $\triangle ABC$ have $\angle A = 36$, $\angle B = 72$, $\angle C = 72$, and draw the bisector $CD$ ($C \in AB$). *Hint: Denote $x := AD$, $y := BD$, express all other lengths in terms of $x$ and $y$, and use trigonometry.*

**(c)** Deduce all of the other values from the table of sines on the previous page (you only need to compute the line of $\sin\theta$, and don't need to mention $0°$ or $90°$).

**Problem 7.** **(a)** For angles $x$ and $y$, show (using the formulas for $\sin(x+y)$ and $\cos(x+y)$) that

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

whenever the expression above is well-defined (don't divide by zero!).

**(b)** Show that in a triangle $\triangle ABC$, with angles $A, B, C$, one has

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

This is also known as the **law of tangents**.