Math Circle Notes Week 8

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§1 Problems from Last Time

Problem 1.1 (Problem 3).

Solution.

1. If $z = 3 + 5i$, $\overline{z} = 3 - 5i$. Notice that we can identify $3 + 5i$ with $(3, 5)$ and $3 - 5i$ with $(3, -5)$, so conjugation corresponds to reflection about the x-axis as we have seen in problem 1(b). In general, the conjugate of $a + bi$ is $a - bi$.

2. $z = -12 - 5i$, so $\overline{z} = -12 + 5i$. Thus

$$\begin{aligned} z\overline{z} &= (-12 - 5i)(-12 + 5i) \\ &= 144 - 60i + 60i + 25 \\ &= 169 \end{aligned}$$

We know that $169 = 12^2 + 5^2$, which is the square of the length of the vector $(-12, 5)$ that z corresponds to. Taking sqaureroot, we see that $\sqrt{z\overline{z}}$ is the length of the vector that z corresponds to. In general, for $z = a + bi$,

$$\begin{aligned} z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 - abi + abi + b^2 \\ &= a^2 + b^2 \end{aligned}$$

and $\sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$.

3. We have seen that for $z = a + bi$

$$z\overline{z} = a^2 + b^2 \geq 0$$

since $a^2, b^2 \geq 0$, and notice that 0 is achieved if and only if $a = b = 0$, which is true if and only if $z = 0$.

4. This problem consists of 2 parts: showing that z' exists, and showing that z' is unique. For existence, we can think of what we did in part (c): $z\overline{z}$ is a positve real number since $z \neq 0$, so

$$z\left(\frac{\overline{z}}{z\overline{z}}\right) = \frac{z\overline{z}}{z\overline{z}} = 1$$

showing that $z' = \frac{\overline{z}}{z\overline{z}}$ would work. We then show uniqueness. Mathematically, the standard way of showing uniqueness is the following: we suppose z' and z'' are both inverses of z, and we show that $z' - z'' = 0$. This means that $z' = z''$, or that any two inverses are equal and hence there is only one unique inverse. I hope you can understand and get used to this philosophy, since it will be useful in mathematical proofs. Coming back to the problem,

$$zz' - zz'' = 1 - 1 = 0$$

so $z(z' - z'') = 0$. We can use either z' or z'' to multiply both sides, say z', to get

$$0 = z'0 = z'z(z' - z'') = z' - z''$$

showing what we wanted to show. Thus the inverse must be unique.

□

In problem 3 we saw that $\sqrt{z\overline{z}}$ makes sense since $z\overline{z}$ is a non-negative real number, and it corresponds to the magnitude of the vector it corresponds to. This motivates the following definition:

Definition 1.2 (Modulus)**.** For a complex number $z = a + bi$, $\sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$ is called the modulus of z, denoted $|z|$.

Remark 1.3. Think about what is the modulus of a real number, in which case the imaginary part bi is 0.

Problem 1.4 (Problem 4a)**.**

Proof. Using the definition we gave above,

$$\begin{aligned} |z_1 z_2| &= \sqrt{z_1 z_2 \overline{z_1 z_2}} \\ &= \sqrt{z_1 \overline{z_1} z_2 \overline{z_2}} \\ &= \sqrt{z_1 \overline{z_1}} \sqrt{z_2 \overline{z_2}} \\ &= |z_1||z_2| \end{aligned}$$

as desired. □

Before doing problem 5, we review polar coordinates in the plane. We all know that every point in the plane can be assigned a coordinate (x, y) where x and y are both real numbers. Polar coordinates give another way to locate points in the plane. Now, we still construct the standard xy axis, but this time for any point in the plane, we locate it with two ingredients: the distance it has to the origin, and the angle it makes with respect to the x-axis. We call this distance r, and call this angle θ, where by convention we use radian as the unit for θ, ranging from 0 to 2π. Then (x, y) can also be denoted (r, θ).

Example 1.5

Consider the point $(1, 1)$ on the plane. Pythogorean theorem tells us that it has maginitude $\sqrt{2}$, and the angle it makes with the x-axis is $\frac{\pi}{4}$ ($45°$). Thus $(1, 1)$ can also be expressed as $(\sqrt{2}, \frac{\pi}{4})$.

You may notice that there is some problem with expressing 0, since the point we refer to is the same whatever number we take in the angle part. For now we don't take care of this issue, and you may choose any sensible number you like for the angle part.

We have the following conversion from polar to usual (cartesian) coordinates, which we state and leave as an exercise if you want to check it:

Proposition 1.6

(r, θ) in polar coordinates corresponds to $(r\cos\theta, r\sin\theta)$ in cartesian coordinates.

Problem 1.7 (Problem 5)**.**

Solution. 1. Note that

$$e^{ix} = e^{0+ix} = e^0(\cos x + i\sin x) = \cos x + i\sin x$$

where we were just using definition 5 in the handout carefully, and hence we have

$$|e^{ix}| = \sqrt{(\cos x)^2 + (\sin x)^2} = 1$$

if we recall how modulus is defined. Geometrically, since e^{ix} has modulus 1, it corresponds to a point on the unit circle (circle of radius 1). Note that e^{ix} corresponds to $(\cos x, \sin x)$ in the plane, so ranging x over $\mathbb{R}$ we can see that the image ranges over the unit circle.

2. Note that $a + bi$ corresponds to the point (a, b) in the plane, and we have just seen that (a, b) can be expressed as (r, θ) using polar coordinates, where r is the magnitude of (a, b) (more precisely just $\sqrt{a^2 + b^2}$) and θ is the angle (a, b) makes with the x-axis. Using the conversion scheme in the above proposition, we see that

$$(a, b) = (r\cos\theta, r\sin\theta) = r(\cos\theta, \sin\theta)$$

This means that

$$a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

as we have proved in part (a).

$\square$

Please pay attention to this form $re^{i\theta}$ we just derived. It will be useful in the next handout.

§2 New Material

Here are some examples of complex polynomials with real coefficients:

Example 2.1
1. $f(z) = z + 1$
2. $f(z) = z^2 + 1$

We can see that there are really no much difference in appearance between real polynomials $f(x) = x + 1$ or $f(x) = x^2 + 1$ except that the input are now complex numbers.

Let's look at the second polynomial $f(z) = z^2 + 1$. If the input is a real number, we know that this polynomial doesn't have any zero, i.e. $z^2 + 1 = 0$ doesn't have any solution in $\mathbb{R}$. However, we may notice that

$$i^2 + 1 = -1 + 1 = 0$$

so $z = i$ is a solution to $z^2 + 1$, meaning that this equation can be solved among complex numbers! This is not a coincidence. Actually, every complex polynomial has a zero, and this is a non-trivial theorem called the **Fundamental Theorem of Algebra**. In problem 1, you will explore some properties the zeros of a polynomial have.