Complex Numbers

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Let's start with a little bit of review of vectors in $\mathbb{R}^2$. We denote a vector in $\mathbb{R}^2$ by $\vec{v}$ and its components by $< x, y >$, where each of x and y is a real number.

1. Compute the following:

   (a) Find the sum of the vectors $< -3, 2 >$ and $< 0, 1 >$

   (b) Give the vector which you get from reflecting the vector $< 3, -4 >$ about the x-axis.

   (c) Given that $\vec{v} =< -4, 6 >$, compute $-3/2 \cdot \vec{v}$

   (d) Calculate $|< -12, -5 >|^2$, where $|\vec{v}|$ is the length of the vector $\vec{v}$ in the usual sense.

   (e) Take the vector $\vec{u} =< \sqrt{3}/2, 1/2 >$ and rotate it counter-clockwise by $\pi/6$ radians. Find the components of the resulting vector. Ask the instructors for help with this problem if you aren't sure what it is asking.

Now we are ready to introduce complex numbers.

Definition 1.
A complex number, often denoted by z, is a number of the form $a+bi$ for $a, b \in \mathbb{R}$. Here i has the property $i^2 = -1$. Complex numbers satisfy the following properties:

- $(a+bi)+(c+di) = (a+c)+(b+d)i$
- $r(a+bi) = (ra)+(rb)i$ if $r \in \mathbb{R}$.
- $(a+bi)(c+di) = (ac+bdi^2)+(ad+bc)i = (ac-bd)+(ad+bc)i$

That is, complex numbers multiply like real numbers with the additional property that $i^2 = -1$.

Complex numbers of the form $a+0i = a$ are called real numbers and $0+bi = bi$ are called imaginary numbers.

2. (a) Find the sum of the complex numbers $-3+2i$ and i.

   (b) Given that $z = -4+6i$, compute $\frac{-3z}{2}$.

   (c) Explain how we can view complex numbers as vectors so that adding two complex numbers is just adding two vectors, and multiplying a complex number by a real number is just multiplying a vector by a real number.

Definition 2.
Given a complex number $z = a+bi$, we define the real part of z, $Re(z) = a$ and the imaginary part of z, $Im(z) = b$. Now we can view complex numbers as sitting on the x-y plane, where the x axis corresponds to the real part and the y axis corresponds to the imaginary part (the coefficient in front of the i). We call this plane the complex plane. Let us attach some geometric meaning to algebraic manipulations.

Definition 3.
Let $z = a+bi$ be a complex number. The conjugate of z, denoted $\bar{z}$, is given by $\bar{z} = a-bi$.

3. (a) If $z = 3 + 5i$, compute $\bar{z}$. What does the operation of conjugation correspond to geometrically? (How are z and $\bar{z}$ related?) Find the conjugate of a general complex number $a + bi$.

(b) If $z = -12 - 5i$, compute $z\bar{z}$. What does this (or maybe the square root of this) correspond to geometrically? Find $z\bar{z}$ where z is now an arbitrary complex number $a + bi$.

(c) Show that $z\bar{z}$ is always a non-negative real number.

(d) Show that if $z \neq 0$, then z has a multiplicative inverse. That is, given $z = a + bi \neq 0$, show that there exists a complex number z' such that $z \cdot z' = 1$. Is this inverse unique?

(e) Take the complex number $u = \sqrt{3}/2 + 1/2i$. Viewing this as a vector, what angle does it make with respect to the real axis (x-axis)? Now find $u^2 = u \cdot u$ and compare this with your answer in problem 1(e). What does this mean geometrically?

Problem 3(b) allows us to define the magnitude (or modulus) of a complex number z.

Definition 4.
The magnitude (also called the modulus) of a complex number z is $|z| = \sqrt{z\bar{z}}$.

4. (a) Show that the magnitude function $|\cdot|: \mathbb{C} \to \mathbb{R}$ is multiplicative. That is, show that $|z_1 z_2| = |z_1||z_2|$ for all $z_1, z_2 \in \mathbb{C}$.

   (b) Suppose $|z| = 1$, explain why multiplying a complex number by z corresponds to a rotation in the complex plane. (Hint: For a complex number $w \in \mathbb{C}$, compare $|zw|$ and $|w|$).

   (c) Fix a positive real number R. Describe geometrically the set of all complex numbers with magnitude less than or equal to R, $\{z \in \mathbb{C} \mid |z| \leq R\}$.

Recall the exponential function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$, where $e = \sum_{n=1}^{\infty} \frac{1}{n!}$ is a constant (approximately 2.718).

We can extend this function to an exponential function on the complex numbers.

Definition 5.
Let $z = x + iy$, where x and y are real numbers. The complex exponential of z, written e^z, is defined by $e^z = e^x(\cos y + i \sin y)$.

5. (a) Let $z = e^{ix}$, where $x \in \mathbb{R}$. Show that $|z| = 1$. What does this mean geometrically? What is the set of points e^{ix} as x ranges over $\mathbb{R}$?

(b) Show that any complex number $a + bi$ $(a, b \in \mathbb{R})$ can be written in the form $re^{i\theta}$ for some $r, \theta \in \mathbb{R}$.

(c) Compute $z_1 z_2$, where $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Use this result to assign a geometric meaning for multiplying two complex numbers.

(d) Use the previous parts to write down a formula for rotating a vector $(x, y) \in \mathbb{R}^2$ counterclockwise by an angle of θ.

6. Use the definition of the complex exponential to:

(a) Show that if $Im(z) = 0$, then e^z agrees with our exponential for real numbers.

(b) Show that the complex conjugate of e^z is equal to $e^{\bar{z}}$.

(c) Show that $e^{2z} = (e^z)^2$. Moreover, show that $e^{z+w} = e^z e^w$. Some trigonometric identities may be useful here.

(d) Find the formulas for $cos3\theta$ and $sin3\theta$ as polynomials in $cos\theta$ and $sin\theta$ using the problem above.

(e) Generalize the previous part to get a formula for $cosn\theta$.

7. Challenge Problems:

   (a) Unlike the complex exponential function, the complex logarithm $ln(\cdot) : \mathbb{C} \to \mathbb{C}$ is not well defined as the inverse of the exponential. What this means is that there is more than one choice of where to send a value. Find all values possible for $ln(1) \in \mathbb{C}$ such that $e^{ln(1)} = 1$.

   (b) Come up with a reasonable definition for complex exponentiation z^x, where $z \in \mathbb{C}, x \in \mathbb{R}$. When z is purely real ($Im(z) = 0$), this should agree with the real exponentiation. (Hint: Polar coordinates may be useful)

   (c) Can you extend your definition in the previous part to one for z^w where now both z and w are complex numbers?

   (d) Using your definition for exponentiation, find all values of i^i.

   (e) As a function of real numbers, $ln(x)$ has the domain $(0, \infty)$. This is not the case when extended to the complex plane. Find all values of $ln(-1)$ (By this I mean all values that would satisfy the property that ln(z) should be the inverse of e^z).