GEOMETRIC COMBINATORICS AND CONVEXITY

LAMC OLYMPIAD GROUP, WEEK 7

A subset S of the plane (e.g. a polygon) is called *convex* iff, given any two points $A, B \in S$, the closed segment $\overline{AB}$ is also contained in S. Another equivalent definition which is often useful is that any line tangent to S (intersecting it in only one point) leaves the whole of S on one side. Convexity is well-behaved under taking intersections, but not under unions.

There is also the notion of a *convex hull* of a set S, which is the intersection of all convex sets containing S. This definition might seem a bit abstract, but the intuition for it is quite simple: one takes a big string and wraps it around S as tight as possible; the contour of the strings and everything inside give the convex hull. The convex hull of a polygon P is always a convex polygon whose vertices are among the vertices of P.

It is a fact that any convex polygon can be broken up into finitely many non-overlapping triangles.[1] In fact, if n is the number of sides of the polygon, one can (only) achieve such a *triangulation* with exactly $n - 2$ triangles.

It is also important know when a polygon is convex in terms of angles. In short, a (closed) polygon is convex if and only if the "internal angles" are all less than or equal to 180°. Also, using the triangulation from before, one has that the sum of all internal angles is $(n-2) \cdot 180°$.

Problem 1. Consider a triangulation of a convex polygon P with $n \geq 4$ vertices into non-overlapping triangles. Color each of these triangles *black* if it has two sides in common with P, *gray* if it has exactly one side in common with P, and *white* if it has no side in common with P. Show that there are exactly two more black triangles than white triangles.

Problem 2. We are given a convex polygon P and two distinct points A, B in its interior.

(a) Show that any four vertices of P form a convex quadrilateral (in some order).

(b) Show that there are four vertices of P forming a convex quadrilateral that contains both A and B (possibly on its sides).

Problem 3. The *diameter* of a polygon is defined as the largest distance between two points lying inside or on the sides of the polygon. Let P be a polygon with diameter d.

(a) Show that there are two vertices of the convex hull of P at distance d.

(b) Suppose that the convex hull of P is a quadrilateral. Show that P has area $\leq \frac{1}{2}d^2$.

Problem 4. You are given a regular hexagon of area 2. What's the maximal area of a triangle that can be inscribed in this hexagon?

Hint: First show that the three vertices of the triangle should lie on the sides of the hexagon.

[1] The boundaries are allowed to overlap, but the interiors are not.

Problem 5. **(a)** Consider 2019 distinct points in the plane, no three collinear, such that every triangle they form has area at most 1. Show that all 2019 points lie inside or on the sides of a triangle of area at most 4 (whose vertices need not be among the initial points).

(b) 5 distinct points lie in the plane, no three collinear, such that every triangle they form has area at most 1. Show that they are all contained inside or on the sides of some *trapezoid* of area at most 3. *Hint: Look at three points forming a triangle with maximal area.*

Problem 6. We are given $2n + 1$ distinct points in the plane ($n \geq 0$), no three collinear. Let A be one of these points.

(a) Show that there is a line ℓ passing through A that partitions the other $2n$ points into two groups of n points (one on each side of ℓ).

(b) Show that there is a circle C passing through A that partitions the other $2n$ points into two groups of n points (one in the interior of C, one in the exterior).