# OLYMPIAD-STYLE PROBLEMS I

COLLECTED FOR THE

LOS ANGELES MATH CIRCLE

**Problem 1** (USAMTS Year 18 - Round 1 - Problem 4 ©Art of Problem Solving Initiative)**.**
Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.

**Problem 2.** Can you draw a path on the surface of Rubik's cube (3x3x3 cube) that goes through every single square on the surface? The path should not go through any vertices.

**Problem 3** (1983 AIME ©MAA)**.**
Find the product of all real solutions to $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$.

**Problem 4.** Let C be a circle with center O. Let A be a point inside of the circle. Consider the set midpoints of all possible chords going through point A. Describe this set (with proof).

**Problem 5** (2004 AIME II Problem 8 ©MAA)**.**
How many positive integer divisors of $2004^{2004}$ are divisible by exactly 2004 positive integers?

**Problem 6** (2002 AMC 12A Problem 16 ©MAA)**.**
Tina randomly selects two distinct numbers from $\{1, 2, 3, 4, 5\}$. Sergio randomly selects one number from the set $\{1, 2, \ldots, 10\}$. What is the probability that Sergio's number is greater than the sum of the two numbers chosen by Tina?

**Problem 7** (2006 AIME I Problem 11 ©MAA)**.**
A collection of 8 cubes consists of one cube with edge-length $k$ for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

(1) Any cube may be the bottom cube in the tower.
(2) The cube immediately on top of a cube with edge-length $k$ must have edge-length at most $k + 2$.

Let $T$ be the number of different towers than can be constructed. What is the remainder when $T$ is divided by 1000?

**Problem 8.** Consider a square of size 102x102 drawn on grid paper. Consider a shape consisting of 101 grid squares total. The shape considered is connected (that is, for any two squares in the shape, there is a chain of squares sharing a side that starts with the first square and ends with the second. Diagonal connections (via corners) are not allowed).

What is the largest integer $n$ such that we are guaranteed to be able to cut $n$ copies of the given shape out of the square of 102x102 (independently of what the shape is)?

*Date*: November 17, 2019.

**Problem 9** (2005 AIME I Problem 12 ©MAA).
For positive integers $n$, let $\tau(n)$ denote the number of positive integer divisors of $n$, including 1 and $n$. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by $S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Let $a$ denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let $b$ denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

**Problem 10** (1988 IMO, proposed by Stephan Beck).
Suppose that $a$ and $b$ are positive integers such that $k = \frac{a^2+b^2}{ab+1}$ is an integer.

Show that $k$ must be a perfect square.

**Hint:** This problem is a somewhat famous example of the power of the *method of infinite descent*, which focuses on contradicting the existence of a "minimal" example or counterexample. For instance, for this problem, suppose toward a contradiction that the result of the problem is false, and let $S$ denote the (then nonempty) set of pairs $(a, b)$ of positive integers such that $k = \frac{a^2+b^2}{ab+1}$ is an integer but not a perfect square.

We can measure the "size" of a given counterexample by the sum $a + b$ and since $\{a + b : (a, b) \in S\}$ is a set of positive integers, it contains a minimal element. That is, we can find a pair $(a, b) \in S$ with the property that $a + b \leq a' + b'$ for any $(a', b') \in S$. To finish the problem, produce a contradiction by producing a strictly smaller counterexample; that is, from this pair $(a, b)$ find some $(a', b') \in S$ with $a' + b' < a + b$.

As an extra hint, we remark that if you relabel as necessary to ensure that $a \geq b$, you will even be able to take $b' = b$ and use the same integer $k$.

For the next two problems, it may be helpful to recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

**Problem 11** (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom). Show that

$$\cos\frac{2\pi}{2n+1} + \cos\frac{4\pi}{2n+1} + \cdots + \cos\frac{2n\pi}{2n+1} = -\frac{1}{2}.$$

**Problem 12** (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom).
Let $A_1A_2\ldots A_n$ be a regular polygon circumscribed by a circle $C$ of radius $r$ and center $O$. If $P$ is a point on that circle, show the sum of the squares of the distances from $P$ to the vertices of the polygon is $2nr^2$.

Show that if instead $P$ is not necessarily on the circle, that the sum of the squares of the distances from $P$ to the vertices of the polygon is $n(r^2 + l^2)$, where $l$ is the distance from $P$ to $O$.

**Problem 13** (2009 AIME II Problem 8 ©MAA).
Dave rolls a fair six-sided die until he gets a six for the first time. Independently, Linda rolls a fair six-sided die until she gets a six for the first time. Find the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die.