INDUCTION II- STIR FRY

BEGINNER CIRCLE 3/3/2013

1. WARM-UP: LEFTOVER PIZZA AND INDUCTION

Domino's, as you may remember, is well known for cutting it's pizza in 2×1 Domino-shaped pieces. In order to compete against Subway, Domino has begun to release the n-footlong. The n-footlong is a pizza that measures 2 by n feet long.

Problem 1. How may ways can Domino's cut a 1-footlong pizza. What about a 2 foot-long? 3 foot-long? What about a 4-footlong? Draw out all the possible cuttings below.

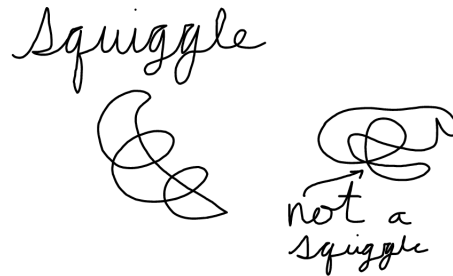**Problem 2.** Recall, the Fibonacci Sequence is the sequence given by the formula

$$F_{n+1} = F_n + F_{n-1}$$

that is, that each number in the sequence is the sum of the two numbers before it. The first few numbers in the sequence are

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

Show that the number of ways to cut the n-footlong is equal to the nth Fibonacci number.

Problem 3. A squiggle is when you take your pen, and draw a closed curve on a piece of paper, and at each crossing is only the crossing of two curves.

Show that every squiggle can be checkerboard colored–that every two regions that share an edge do not have the same color.

by induction on the number of crossings in the squiggle, n.

(i) First write down what $P(n)$ is in full sentences.

(ii) Write down $P(1)$, and explain why it is true.

(iii) Show that $P(n)$ implies $P(n+1)$. (Hint: is there a way to take a $n+1$-squiggle and turn it into a n-crossing squiggle in a way that preserves a checkerboard pattern?)

2. Induction in Sequences

Sequences are one of the most common places to find induction problems. When applying induction to the below problems, we will usually induct on the number n, which denotes the placement in the sequence.

Problem 4. The sequence of square numbers is

$$1, 4, 9, 16, \ldots$$

Prove that the nth square number is

$$n^2 = 1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1),$$

that is, the sum of the first n odd numbers.

Problem 5. The sequence of triangle numbers is

$$1, 3, 6, 10, 15, 21, \ldots$$

where the nth triangle number is equal to

$$t_n = 1 + 2 + 3 + \ldots + n$$

Show that the nth triangular number is equal to

$$1 + 2 + 3 + \ldots n = \frac{n(n + 1)}{2}$$

Problem 6. The sequence of powers of 2 is given by 2^n, and is given by

$$1, 2, 4, 8, 16, 32 \ldots$$

Show that the sum of the first n numbers of this sequence is given by

$$1 + 2 + 4 + \ldots 2^n = 2^{n+1} - 1$$

Problem 7. If we draw n lines in the plane, such that no two lines are parallel and no three lines intersect at a single point, show that the number of regions that the plane is divided into is

$$\frac{n(n + 1)}{2} + 1$$

3. INDUCTION IN COMBINATORICS

Combinatorics is another place where we find a lot of induction. We've already seen a few proofs that use induction in Combinatorics, but we didn't call it that. Usually, in Combinatorics, we induct on the number of objects being arranged/combined.

Problem 8. Show that the number of ways to rearrange n items is

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

Problem 9. Show that the number of ways to pick a group (of any size) of kids from a classroom that is n students large is 2^n

Problem 10. Using the rule that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

show that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Problem 11. Using the rule that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

show the Hockey Stick Rule:

$$\binom{n}{m-1} = \binom{n-1}{m-1} + \binom{n-2}{m-2} + \binom{n-3}{m-3} + \ldots + \binom{n-m+1}{0}$$

4. INDUCTION IN GRAPH THEORY

Induction is also used in graph in theory. Usually, we will induct on the number of edges in a graph, or the number of vertices in a graph.

Problem 12. Let G be a planar graph. Then let V be the number of vertices, E be the number of edges, and F be the number of faces. Then

$$V - E + F = 2$$

(Hint: Induct on $n = V + E$)

Problem 13. Let G be a graph with n vertices. Then there are two vertices, a or b, that you can remove from the graph G and still G be connected.

Problem 14. Show that every connected graph has at least $n - 1$ edges.

Problem 15. Recall, a *coloring* of a graph is a way to assign colors to the vertices of a graph so that no two vertices that share an edge have the same color. Suppose that we know that every planar graph contains a vertex of degree at most 6. Show that every planar graph can be colored with 6 colors (Hint: Induct on the number of vertices)

Problem 16. Recall, an Eulerian cycle is one that goes through every edge once. Show that if every vertex of a graph has even degree, then the graph contains an Eulerian cycle. (Hint: Induct on the number of edges.)
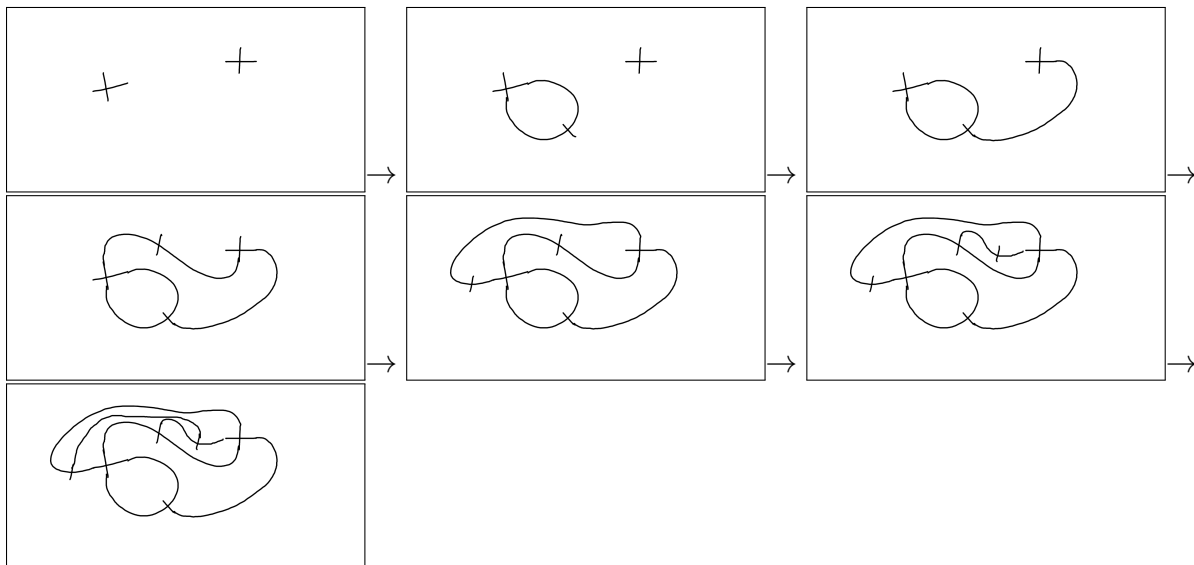
5. INDUCTION IN GAMES

In combinatorial game theory, we frequently use induction to prove that a certain position is a winning position for a game.

Problem 17. Let us run over the rules for the Debate game. In the debate game, are given turns to present arguments that are between 1 and 5 minutes in length. The person that gets the final word in wins the game. Prove that if the game starts at a multiple of 5 minutes, then the first player wins the game, otherwise, the second player wins the game.

Problem 18. The rules of Soy sprouts are as follows: We start with some stars drawn on a sheet of paper. The tips of the crosses are called spikes.

- On your turn, you may connect two spikes with an edge. This edge may not cross any edge already drawn. When you have drawn an edge, you may, **if you wish**, add a short stroke across that edge to make two more spikes.
- Each spike may take at most 1 edge
- When you cannot place an edge on your turn, you lose.

When you start with a single star with n spikes, show that the first player has a winning strategy if n is a multiple of 4 plus 1

INDUCTION SHEET: USE ONE PER PROBLEM!

Define the Problem in terms of a statement $P(n)$**.**

Prove the statement $P(1)$**.**

Show that $P(n) \to P(n+1)$**.**