Lesson 5: More remainders and divisibility.

Konstantin Miagkov

May 5, 2019

Problem 1.
a) Show that a number is divisible by 2 if and only if its last digit is even.

Proof. Suppose r is the last digit of a positive integer n. Then $n = 10k + r$ for some positive integer n. Since $10k$ is always even, n is even if and only if r is. □

b) Show that a number is divisible by 4 if and only if its last two digits make a number divisible by 4.

Proof. If the last two digits make a number r, then $n = 100k + r$. Since $4 \mid 100$, n is divisible by 4 if and only if r is. □

c) Can you generalize these principles to make a divisibility criterion for any 2^n?

Proof. m is divisible by 2^n if and only if the number made by the last n digits of m are divisible by 2^n. Proof is similar to parts a) and b). □

d) Can you do the same for 5^n?

Problem 2.
a) A positive integer n has remainder 7 when divided by 9. Can it have remainder 2 when divided by 3?

Proof. A positive integer n that has remainder 7 when divided by 9 will have $n = 9m+7$ where m is a nonnegative integer. Therefore when being divided by 3, $n = 9m + 7$ will have the same remainder as 7 since $9m$ is divisible by 3. 7 has remainder 1, so clearly n cannot have remainder 2 when being divided by 3. □

b) A positive integer n has remainder 23 when divided by 144. Can it have remainder 29 when divided by 90?

Proof. Suppose it can, $n = 144m+23 = 90k+29$ where k and m are nonnegative integers. Consider the remainder of n by 9. When being divided by 9, $n = 144m + 23$ will have the same remainder as 23 since $144m$ is divisible by 9. 23 has remainder 5 when divided by 9. With similar argument, $n = 90k + 29$ will have the same remainder as 29 when divided by 9. 29 has remainder 2 when divided by 9, which is a contradiction. □

Problem 3.
A positive integer n has remainder 2 when divided by 3 and remainder 9 when divided by 11. What will be its remainder when divided by 33? Find all possible answers and show that none other exist.

Proof. $n = 11m + 9$ where m is a nonnegative integer. We will look at the remainder of n by 33 depending on the remainder of m by 3.

Case 1: m is divisible by 3, i.e. $m = 3k$ where k is a nonnegative integer. Then $11m = 33k$ will be divisible by 33, so n will have remainder 9 by 33.

Case 2: m has remainder 1 when divided by 3, i.e. $m = 3k + 1$ where k is a positive integer. Then $n = 11(3k+1)+9 = 33k+20$ will have remainder 20 when divided by 33.

Case 3: m has remainder 2 when divided by 3, i.e. $m = 3k + 1$ where k is a nonnegative integer. Then $n = 11(3k+2)+9 = 33k+31$ will have remainder 31 when divided by 33.

We can conclude that 9,20,31 are the only possible remainders. But we also have that n has remainder 2 when divided by 3. Since 33 is divisible by 3, the remainder when divided by 3 will be the same as remainder when divided by 33. In the above three cases, the remainders are 9, 20 and 31. The only one has remainder 2 when divided by 3 is $20 = 3 \times 6 + 2$. So 20 is the only possible remainder of $n/33$ □

Problem 4.
a) How many zeros does the number 10! end with? Reminder: $n!$ reads n *factorial* and equals $1 \cdot 2 \cdot \ldots \cdot n$.

Proof. Note that if 5^a and 2^a both divide some number n, then so does $2^a \cdot 5^a = 10^a$ because 2 and 5 are prime and thus there must be at least a 2's and 5's in n's prime factorization. Also, if n ends with a zeroes, then $10^a \mid n$ and thus 2^a and 5^a both divide n. So we just need to find the largest a such that 2^a and 5^a both divide $n!$.
Again, since 2 and 5 are both prime, we just need to calculate the number of times they occur in the prime factorization of $n!$. Since $10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot 2 \cdot 3 \cdot 7 \cdot 2^3 \cdot 3^2 \cdot 2 \cdot 5 = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, two is the smallest number such that both 2^2 and 5^2 divide $n!$, so $n!$ ends in two zeroes. □

b) Same question for 100!

Proof. We can proceed like in part a), except instead of prime factorizing every number between one and a hundred (although this is a possible solution), we can just count the number of multiples of 5 in the range of 1 to 100 inclusive, and then add it to the number of those multiples which are multiplied by another 5, and add it to the number of those multiples that are multiplied by another 5 and so on. Except since $5^3 = 125 > 100$, we just need to check the first two cases (numbers divisible by 5 and numbers divisible by 25).
Note that we don't need to find the largest power of 2 which divides 100! since two appears as a prime factor of numbers in the interval [1,100] more often than 5. Finally, since 5^1 goes into 100 twenty times, and 5^2 goes into 100 four times, the highest power of 5 such that 5^a divides 100! must be $20 + 4 = 24$. □

Problem 5.

Over the last two weeks we have seen the divisibility criterion for powers of 2, as well as for 3, 9 and 11. Find a divisibility criterion for some other odd integer $n > 1$ of your choosing. The only other restriction is that your chosen n should not be divisible by 5.