INEQUALITIES WITH INTEGERS

LAMC OLYMPIAD GROUP, WEEK 5

You've had some practice with proving inequalities of real numbers (using $QM \geq AM \geq GM \geq HM$, Cauchy-Schwarz), but the tricks used can change when we're dealing with integers. In the problems below, try to exploit factorizations, divisibility, and the fact that some expressions grow much quicker than others (e.g., 2^n compared to n^2).

For a real number x, define its ***floor*** $\lfloor x \rfloor$ to be the largest integer $\leq x$, and its ***ceiling*** $\lceil x \rceil$ to be the smallest integer $\geq x$. For example, $\lfloor -0.42 \rfloor = -1$ and $\lceil -0.42 \rceil = 0$.

Problem 1. **(a)** Show that for all non-negative integers n the inequality $2^n \geq 1+n$ holds.

(b) Show that for all non-negative integers n the inequality $2^n \geq 1 + n + \frac{n(n-1)}{2}$ holds.

(*c) Show that for any positive integer k, there exists some value $n \geq 2$ such that $2^n > n^k$.

Problem 2. Find all positive integers n such that $n^2 + 1 \mid n^3 + 2n + 1$.

Problem 3. Find all integers n satisfying $n^3 - 17n + 26 > 0$.

Problem 4. Find all pairs of positive integers (m, n) satisfying the inequality

$$mn + 3m - 17n \leq 55.$$

Problem 5. **(a)** Find all positive integers n such that $2^n + 3^n = 5^n$.

(b) Find all positive integers n such that $3^n + 4^n = 5^n$.

Problem 6. **(a)** Let $x > 0$ be a positive real number. Suppose that for all positive integers n, one has $n \mid \lfloor nx \rfloor$. Show that x is an integer.

(b) Let $x, y > 0$ be positive real numbers. Suppose that for all positive integers n, one has $n \mid \lfloor nx \rfloor \lfloor ny \rfloor$. Show that at least one of x and y is an integer. *Hint: Look at primes.*

Problem 7. Find all positive integers n such that $n! + 1 \mid (n+1)! \cdot (n+1)$.

Problem 8. Find all positive integers a, b such that $a^2 + b^3 = 7ab$.

Problem 9. Let a, b be positive integers such that $3^a - 2^a \mid 3^b - 2^b$. Show that $a \mid b$.

Problem 10. Let n be a positive integer.

(a) Show that for $n \geq 3$, one has $\left(1 + \frac{1}{n}\right)^n < n$. *Hint: Try a telescoping product.*

(b) Show that $\sqrt[n]{n} > \sqrt[n+1]{n+1}$ for $n \geq 3$.

(c) Let $a, b \geq 3$ be positive integers. When is $a^b < b^a$?

Problem 11. For a positive integer k, let $\sigma(k) = \sum_{d|k} d$ be the sum of its (positive integer) divisors; for example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. Show that

$$\sigma(1) + \sigma(2) + \cdots + \sigma(n) \leq n^2.$$

Problem *12. Show that

$$\sum_{x=1}^{N} \sum_{y=1}^{N} \left\lfloor \frac{4xy}{x^2 + y^2} \right\rfloor \leq 2N + N(N+1)\sqrt{3}$$

for all positive integers N.