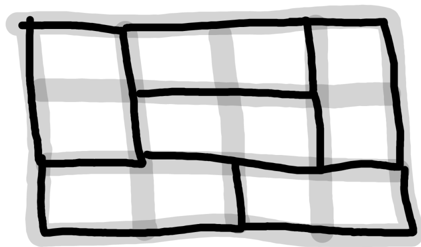
INDUCTION AND PIZZA

BEGINNER CIRCLE 2/24/2013

1. Triominoes Pizza

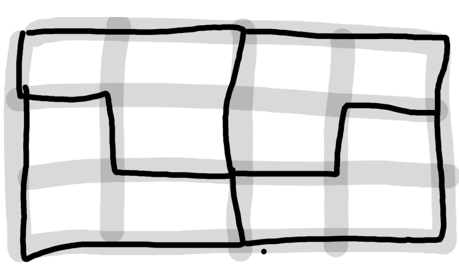
I am sure that you are all aware of Domino's Pizza, whose logo is a 2×1 rectangle:

The reason that Domino's Pizza is called Dominos is because they always cut their pizzas into 2×1 rectangles, like shown:



But do you know about Triominoes? They are much less popular than Domino's Pizza. They look like dominoes, but have an additional square.

Triomino's Pizza cuts their pizza's into Triominos,



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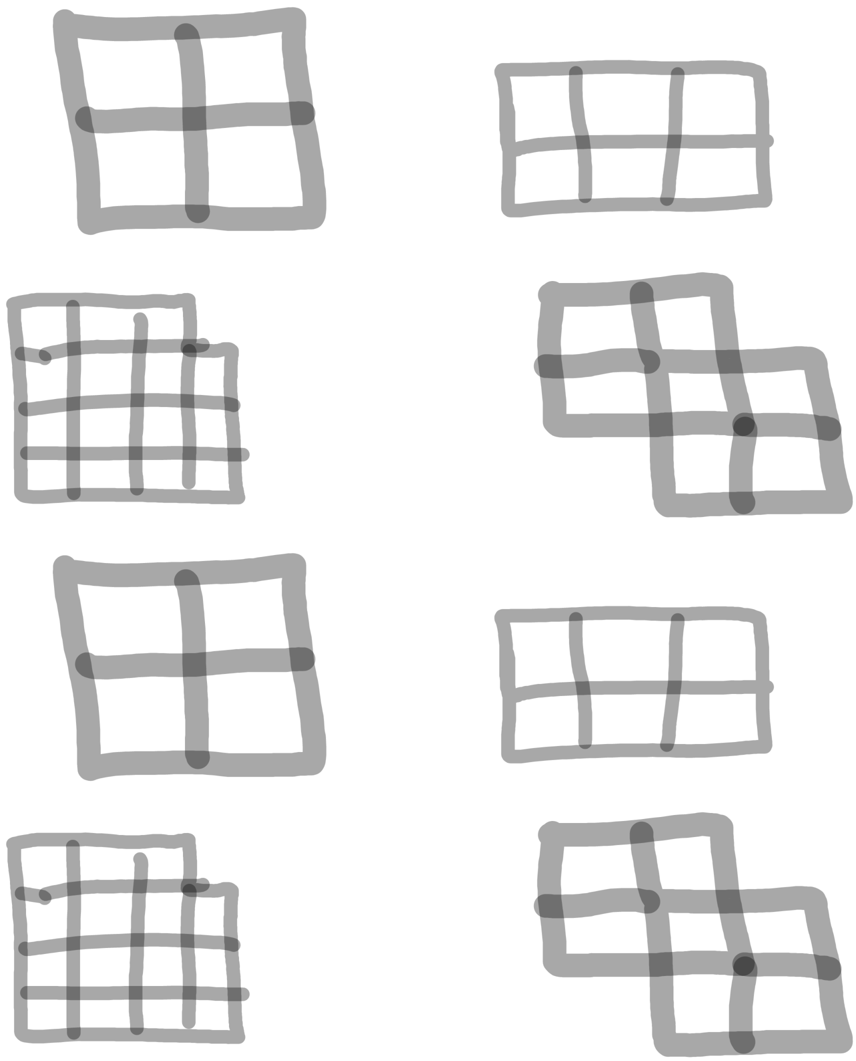
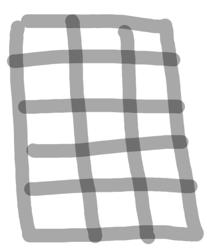
Problem 1. Which of the following pizza's could be served by Domino's? What about by Triomino's? (Remember, a pizza can only be served by Domino's if it can be cut into Domino shaped slices; likewise, a pizza can only be served by Triomino's if it can be cut into Triomino sliced pieces.)

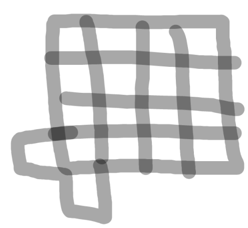


Problem 2. Can you come up with a pizza shape that has area divisible by 3, but cannot be sold by Triomino's pizza?

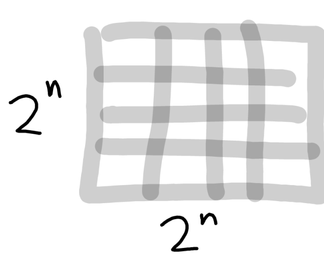
Problem 3. Can you come up with a pizza whose area is divisible by 2, but cannot be sold by Domino's Pizza?

Problem 4. Check if these pizza's can be served by Domino's. What about Triomino's?



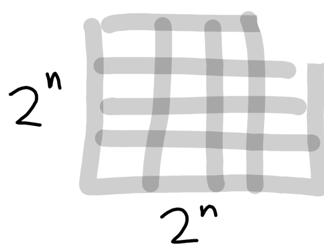


Problem 5. To celebrate their 50th anniversary, Domino's releases a new line of pizza, called the "4^n pizzas". These pizzas measure 2^n by 2^n along a side. Show that the 4^1, 4^2, and 4^3 pizzas can be sold by Domino's. (Draw out all of the pizzas)



Problem 6. Assuming that the "4^{n-1} pizza" can be sold by Domino's, can you show that a "4^n pizza" can be served by Domino's? Use full sentences. (Hint: Break the "4^n pizza" into 4 smaller pizzas that you know can be served by Domino's)

Problem 7. Not to be outdone by Domino's, Triomino's releases the "$4^n - 1$ pizza" , which look just like the "4^n pizzas" but are missing a piece in the corner. Verify that the $4^1 - 1$, $4^2 - 1$ and $4^3 - 1$ pizzas can all be sold by Triomino's. (Draw out all of the pizzas)



Problem 8. Assuming that the "$4^{n-1} - 1$ pizza" can be sold by Triomino's, show that the "$4^n - 1$ pizza" can be served by Triomino's. Use full sentences. (Hint: Try doing what you did for the Domino's Pizza. Why doesn't it work? How can you fix it?)

2. Induction

One of the most powerful techniques in mathematics is the method of induction.
Induction works on the following principle: Let $P(n)$ be a logical statement that depends on the number n.
Suppose that we know that $P(1)$ is true.
Also, suppose that whenever $P(k)$ is true, then $P(k+1)$ is also true
.
Then it is the case that the statement $P(n)$ is true for every n.
Probably the easiest way to learn about induction is to see an example. Here is a very simple induction problem

Problem 9. Morgan likes pizza a lot. Morgan can always eat another slice of pizza– that is, if Morgan can eat a pizza with 5 slices, he has no problem with eating a pizza with 6 slices. Additionally, we know that Morgan has no problem eating a single slice of pizza. Show that Morgan can eat pizzas of arbritary size.

The statement $P(n)$ is then

$$P(n) = \text{ Morgan can eat Pizzas with } n \text{ slices}$$

We know additionally that

$$P(1) = \text{ Morgan can eat pizzas with 1 slice}$$

is also true, and additionally

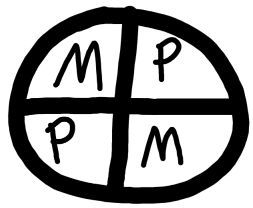
$$P(k) \Rightarrow P(k{+}1) = \text{ If Morgan can eat a pizza with } k \text{ slices, he can eat one with } k+1 \text{ slices}$$

is also true.
It logically follows by the principle of induction, that Morgan can eat a pizza with any number of slices; that is the statement $P(n)$ is true for every n. Let us look at a more

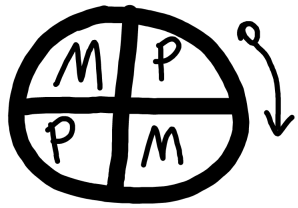
difficult problem.

Problem 10. Derek is about to eat a pizza. The pizza has an even number of slices, and half the slices have pepperoni, and the other half the slices have mushrooms. The are randomly assorted. Derek likes pepperoni, and dislikes mushrooms. The way Derek eats pizza is like this: He starts by eating a slice of pizza, and then eats the slice clockwise of that one, and then the slice clockwise of that one, and so on until the pizza is finished. Derek is ok with eating a slice with mushrooms, as long as at any given point he has eaten at least as many slices of pepperoni as slices with mushrooms. Why is it that Derek can happily eat his pizza?

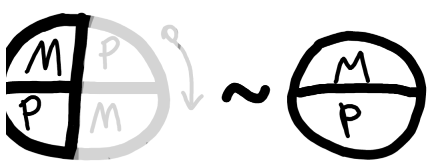
First, let's look at an example. Here is a pizza with four slices.



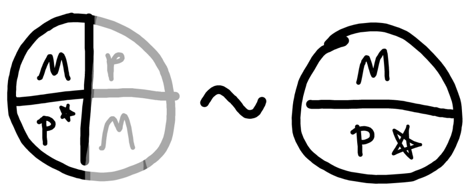
If Derek starts at the marked piece, then he will be able to eat his pizza happily in a clockwise direction.



Let's now prove that every pizza with 4 slices can be enjoyed by Derek.
Fact: There always is a mushroom slice that lies clockwise and next to a pepperoni slice.
If we remove these two adjacent slices, we get something that looks like a pizza with 2 slices, and clearly every pizza with 2 slices can be enjoyed by Derek.



Let's mark the slice where Derek must start in order to enjoy his 2-slice pizza



Now what happens if we add the two slices back in? Because the mushroom slice lies clockwise of the pepperoni slice, Derek will eat the pepperoni slice first and then the mushroom slice. Derek can still enjoy his pizza with the two slices back in.
Can we extend this process to bigger and bigger pizzas? Let us fit this in the language of induction:

Problem 11. To solve this problem, we need to phrase it in a logical statement that depends on a number n.

(i) What is the statement $P(n)$ that we are trying to prove. Write out your statement in a complete statement.

(ii) The next part of induction is to show that the initial statement, $P(1)$ is true. Write the initial statement $P(1)$ in full sentences, and explain why it is true.

(iii) The next part of induction is to prove the "inductive step" that is
if $P(k)$ is true, then $P(k+1)$ is true
Can you write the explain what the above statement is in full sentences for this problem?

This is the tricky statement to prove, but it is the one that involves removing

the 2 special slices, and adding them back in.

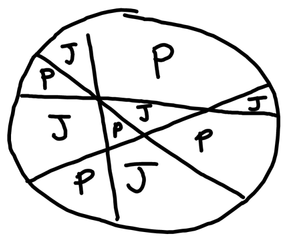
(iv) Explain why

$P(k)$ then $P(k+1) =$ If Derek enjoys every pizza with $2k$ slices, then he can enjoy every pizza with $2k+1$ slices

is true in detailed sentences.

Round Table Pizza works with round pizzas, and cuts them with a pizza cutter in straight lines. Their specialty pizza is the "Pineapple Jalapeno Checker Pizza", where

(i) Every slice is either Pineapple or Jalapeno
(ii) The only slices that share an edge with a Jalapeno slice are Pineapple slices, and the only slice that share an edge with a Pineapple slice are Jalapeno slices



Show that no matter how Round Table Pizza cuts their pizzas, they can checker the Pineapple and Jalapeno slices.

Problem 12. What is the inductive statement $P(n)$? Hint: n will denote the number of slices made

Problem 13. What is $P(1)$, and verify that it is true.

Problem 14. Why is it that if we know $P(k)$ is true, then we know that $P(k+1)$ is also true? (Hint: For starters, try showing that if $P(1)$ is true, then $P(2)$ is also true.)

Problem 15. Conclude that $P(n)$ is true for all n

3. Additional Induction Problems

Problem 16. Show (using induction) that

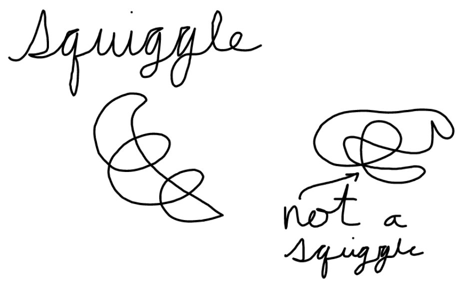
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

(i) Write out the statement $P(n)$?

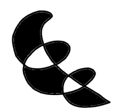
(ii) What is $P(1)$ say, and why is it true?

(iii) Prove that if $P(n)$ is true, then $P(n+1)$ is true.

Problem 17. A squiggle is when you take your pen, and draw a closed curve on a piece of paper, and at each crossing is only the crossing of two curves.



Show that every squiggle can be checkerboard colored–that every two regions that share an edge do not have the same color.



by induction on the number of crossings in the squiggle, n.

(i) First write down what $P(n)$ is in full sentences.
(ii) Write down $P(1)$, and explain why it is true.
(iii) Show that $P(n)$ implies $P(n+1)$. (Hint: is there a way to take a $n+1$-squiggle and turn it into a n-crossing squiggle in a way that preserves a checkerboard pattern?)