NEGATIONS

BEGINNER CIRCLE 10/27/2012

1. Warm Up: Opposite Day

Today is Opposite day. Isaac and Jeff decide to be very clever and hold a conversation only saying the exact opposite of what they mean. Can you translate what they are saying without just adding the word ”not” in front of the sentence? For example, if Jeff said “All swans are white” then what he really means is “There exists a swan that is not white”

Isaac: My day is anything but good.

Jeff: There is a cloud outside!

Isaac: I have done only 1 problem on my homework!

Jeff: I have done all of the problems on my homework.

Isaac: I think that outside is not 75 degrees.

Jeff Hicks/Los Angeles Math Circle/UCLA Department of Mathematics .

Jeff: Nobody will show up to math circle today.

Isaac: Way! I agree. There will be at least one person that is missing.

Jeff: If not a single person shows up, I will teach!

Isaac: If today is Tuesday, I will go to the store!

Now have a conversation with your neighbor!

2. General Statements

Let us talk about general statements. A general statement is a fact about a lot of objects: for instance

> " Every Math Circle instructor is over 4 feet tall "

is a statement about *all* Math circle instructors.

Problem 1. Suppose we found a person that was only 3 feet tall. Then what could we say about them?

A different type of general statement tells us about the existence of certain things, like

> " Every Math Circle classroom contains an instructor "

This statement says that we can always find a certain type of object within a set.

Problem 2. Suppose we found a classroom without an instructor. What could we say about the classroom?

Sometimes the wording of a statement can be tricky, especially with the word or. Take for instance, the statement,

> " In order to be a Math Circle instructor, you must like teaching or be handsome. "

However, this statement doesn't mean that all of our Math Circle instructors $either$ like teaching or are handsome: take for instance Derek, who is both of these things. When we write "A or B", we mean "A or B or $both$".

3. Negations

What do we mean when we say the opposite of a statement? Let us look at an example of a negation:

“ A = I like every kind of pie. ”

How do we properly negate this sentence?

“ (not A) = There is a kind of pie that I don’t like. ”

A negation of a statement is true if and only if the original statement is false. If we take a negation of a negation, we should get our original statement back. For instance

“ (not not A) = There is no kind of pie that I don’t like ”

is equivalent to the statement that “I Like every kind of pie”

Problem 3. Negations can be a bit tricky to figure out! Sometimes, we may have a statement that looks like a negation, but really isn’t. For instance, look at this statements:

“ B= I like no kind of pie ”

Why is this not a negation of “I like every kind of pie”

Problem 4. Write down a negation for the following statements:

(1) Derek’s favorite color is Blue

(2) Isaac likes every color except red.

(3) Every number is even.

(4) There is a letter of the alphabet written with just one pen stroke.

(5) In order for it to be hot outside, it can be sunny or it can be summer

Problem 5. Match each statement to the one which is the proper negation of it

Every Turtle is slow	There are turtles that are not slow and not fast
There is a fast turtle	There exists a turtle that is not slow
Every Turtle is either fast or slow	All turtles are either not fast, or not slow
There is a turtle that is fast and slow	Every turtle is not fast

Problem 6. Let us look at general statements of the type

“ Every A has property B”

These are statements like

Every instructor in Math Circle has a Beard

. Compare this to the statement

There is a Math Circle instructor that has no beard

(a) Explain why if the first statement is true, then the second statement is false.

(b) Explain why if the second statement is true, the first statement is false.

(c) Conclude that the first statement is the negation of the second statement.

Problem 7. Using the previous problem as a hint, explain why the opposite of “Every A has property B” is “There is a A that does not have property B”

4. Contrapositive

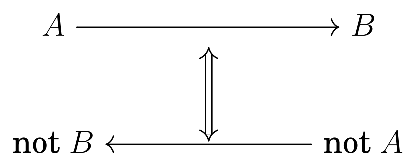
Consider the statement

> " If it is raining outside, then Derek will open his an umbrella. "

This is a "if A, then B" type of statement. The rule of contrapositive says that this is the same thing as "if not B, then not A." How can we rephrase this top sentence?

> " If Derek is not holding an umbrella, then it is not raining. "

We call this reversed statement a contrapositive. A good way to remember contrapositive is with a picture



Problem 8. Explain why the statement "If you are Math Circle instructor, then you are older than 18" is logically equivalent to "If you are not yet 18, you are not a Math Circle instructor"

Problem 9. Explain why "If A is true, then B is true" is logically equivalent to "If B not true, then A is not true"

Problem 10. Can you take the contrapositive of the following statements?

(1) If it is sunny, then it is not raining

(2) If Every swan is white, then no swan is black

(3) If it is too late, then you should go to sleep

(4) If you don't eat food,then you'll become hungry.

Proving the contrapositive of a statement can sometimes be easier than proving the statement itself. For instance let us try to prove

" If x^2 is even, then x is even"

Problem 11. (A proof using the contrapositive)

(a) What is the contrapositive of the statement "If x^2 is even, then x is even"

(b) Use $\mod 2$ arithmetic to prove that the square of an odd number is odd.

(c) Conclude that if x^2 is even, then x is even.

5. Negation of And and Or

How do we negate a statement like

“ Today is Cold and Rainy ”

To help us understand this statement, we use a **truth table**.

Problem 12. Fill out the truth table

Today is Cold	Today is Rainy	Is Today is Cold and Rainy?
T	T	
F	T	
T	F	
F	F	

What should the negation of this statement be? Well, let us look:

Problem 13. Fill out the truth table

Today is Cold	Today is Rainy	It is not the case that Today is Cold and Rainy?
T	T	
F	T	
T	F	
F	F	

Problem 14. Finish filling out these truth tables:

Today is Cold	Today is Rainy	It is not Cold and it is not Rainy
T	T	
F	T	
T	F	
F	F	

Today is Cold	Today is Rainy	It is not Cold or it is not Rainy
T	T	
F	T	
T	F	
F	F	

Why do these tables show that the proper negation of

“ A and B”

is the statement

“Not A or Not B?”

6. Proof by Contradiction 1

Sometimes in order to prove something, it is easier to prove that it cannot be false. Why wouldn't something be false? The best way to see this is by example. Let us prove that the square root of 2 is irrational.

Recall that a number is called irrational if it is not expressible as a fraction. How will we prove that $\sqrt{2}$ is not rational? We will assume that is expressible as a fraction, and prove that it can't be expressible as a fraction. Since it can't be true that $\sqrt{2}$ is both a fraction and not a fraction, our original assumption must be incorrect.

Problem 15. (Proof that $\sqrt{2}$ is irrational)

(a) Assume (for our contradiction) that $\sqrt{2} = \frac{a}{b}$. Why is it that we can choose a and b so that only one of them is divisible by 2?

(b) We then have that $2 = \frac{a^2}{b^2}$. Can you prove that a is even?

(c) Why does 4 divide a^2?

(d) If 4 divides a^2, how do we know that 2 divides b^2?

(e) If 2 divides b^2, is b even or odd?

(f) Why is this a contradiction? Conclude that there is no a and b so that $\frac{a}{b} = \sqrt{2}$.

Problem 16. Why doesn't this proof work when we try to show that the square root of 4 is not rational?

Problem 17. Can you use the same kind of proof to show that the square root of 3 is not rational?

7. Proof by Contradiction 2

One of the classic proofs of mathematics is that there is an infinite number of prime numbers. While it is hard to show that there is an infinite number of primes, it is easy to show that it is not possible for there to be a finite number of primes. We will use a proof by contradiction to show that this is true.

Problem 18. (Proof of infinitely many primes)

(a) If we want to prove that there are infinitely many primes, what should we instead assume for out contradiction?

(b) With our assumption, why is it that we can list all of the primes,

$$p_1, p_2, p_3 \dots p_n$$

(c) Why is it that p_1 does not divide $p_1 \times p_2 \times p_3 \times \dots \times p_n + 1$. (Hint: look at the remainder)

(d) Why is it that none of the $p_1, p_2, \dots p_n$ divide $p_1 \times p_2 \times p_3 \times \dots \times p_n + 1$?

(e) What can we conclude about $p_1 \times p_2 \times p_3 \times \dots \times p_n + 1$? How is this a contradiction?

Problem 19. Suppose that we know that $p_1, p_2, p_3 \dots p_n$ are prime numbers. Does this prove that $p_1 \times p_2 \times p_3 \times \dots p_n + 1$ is a prime number? Why or why not?

8. Paradoxes

Sometimes it can be hard to determine if a statement is true or false. In fact, sometimes, it is impossible to tell if a statement is true or false. Take for instance the statement

> " This statement is false "

This seems troublesome. Why?

Here is a problem (from Clint) that is particularly interesting:

Problem 20. Sam is very pleased with his most recent purchase, a shiny book called the *Complete Non-Self-Reference Reference* (CNSRR). The ads tell us that this book mentions exactly all the books that doesn't mention themselves. For instance, *Harry Potter and the Sorcerer's Stone* doesn't mention itself, as a book, anywhere in the text, so CNSRR mentions it. On the other hand, the dictionary does mention itself, so it does not get mentioned in CNSRR. When Edanel hears about Sam's new purchase, he gets worried. "I think there's a problem with your new book," he says. What does Edanel mean? What's the problem?

Problem 21. Pablo, Gary, Sara, Mark, and Wanda were playing clue last week. Each one was playing as one of five characters: Professor Plum, Mr. Green, Miss. Scarlet, Colonel Mustard or Mrs. White. Everybody but one person committed a crime in a certain room with a certain weapon. You interview the suspects, but quickly realize that they only tell lies. Using these clues, can you solve the mystery?

Interview Transcript

(1) Pablo: There is a person who is not one of the following
   (a) Pablo
   (b) The woman who is playing Colonel Mustard
   (c) The Man who used Wrench
   (d) the Kitchen criminal
   (e) The one who is innocent.
(2) Gary: At least one player's name did start with the same letter of the character they were playing.
(3) Sara: Wanda did not steal the knife
(4) Mark: The person who isn't Gary or the person who isn't Sara were not in the Ballroom and not in Study
(5) Wanda: Mrs. White did not use the Lead Pipe or Professor Plum did not use the Candlestick
(6) Pablo: The murderer did not have the rope
(7) Gary: Mark was not seen in the Cellar with the Candlestick
(8) Sara: Whoever broke the wrench did not do it in the Ballroom.