Triangulated Polygons and Frieze Patterns

LA Math Circle *

1 Symmetries of Frieze Patterns

First, a lesson in architecture. A *frieze* is a horizontal band of decoration around a building. These are some friezes from buildings in Ancient Persepolis. Note how they feature symmetrical, repeating patterns, and it is easy to see how you could keep drawing them to the left or the right, infinitely far. In mathematics, a *frieze pattern* will be a pattern drawn on an infinitely long band (if you want some kind of precision, imagine the set $\mathbb{R} \times [0, 1]$, which is the set of points (x, y) in the plane where $0 \leq y \leq 1$), that has a certain kind of symmetry.

Figure 1: Friezes from Ancient Persepolis

The symmetry groups of frieze patterns are called *frieze groups*, and there are only 7 of them (up to isomorphism). They are given by the following examples using common symbols:

LLLLLLLL
LΓLΓLΓLΓ
VVVVVVVV
SSSSSSSS
VΛVΛVΛVΛ
BBBBBBBB
HHHHHHHH

Problem 1 These symmetry groups include the following types of symmetries: horizontal reflections (H), vertical reflections (V), (horizontal) translations (T), (180°) rotations (R), and glide reflections (G). Using those abbreviations, can you identify which types of symmetry are present in each of the 7 frieze groups?

Problem 2 Which symmetry group(s) do the frieze patterns in Figure 1 have?

*Adapted by Aaron Anderson from Conway and Coxeter

Problem 3 Can you classify the symmetry groups of the following tables of Roman numerals:

(a)

…	I	I	I	I	…

(b)

	I		I		I		I		…
…		I		I		I		I	

(c)

	I		I		I		I		…
…		I		II		I		II	
	I		I		I		I		…

(d)

	I		I		I		I		I		I		
…		III		I		II		II		I		III	
	II		II		I		III		I		II		…
…		I		I		I		I		I		I	

Figure 2: Frieze Patterns of Roman Numerals

2 Unimodular Frieze Patterns of Numbers

In the last problem, the Roman numerals I, II, III all conveniently had lots of symmetry. However, as this is a *math* circle, we'd like to study the patterns of numbers, and we care more about arithmetic properties than the way the numbers are drawn, so we'll use regular Arabic numerals, and just look at the symmetries of their arrangement.

Our new, numerical frieze patterns will be a pattern made of numbers, arranged in (infinite) rows, with each row staggered a bit from the last. The first (top) row will be all 1s, as will the last row. We fill the rows in between with any numbers, which are positive unless specified, subject to one more constraint, which we'll get to in a moment.

	1		1		1		1		1		1	
…		1		3		1		3		1		…
	2		2		2		2		2		2	
…		3		1		3		1		3		…
	1		1		1		1		1		1	

	1		1		1		1		1		1	
…		1		2		3		1		2		…
	2		1		5		2		1		5	
…		1		2		3		1		2		…
	1		1		1		1		1		1	

	1		1		1		1		1		1		1		1	
…		1		3		2		2		1		4		2		…
	1		2		5		3		1		3		7		1	
…		1		3		7		1		2		5		3		…
	2		1		4		2		1		3		2		2	
…		1		1		1		1		1		1		1		…

Figure 3: Frieze Patterns of Numbers

We will also define the *order* of a frieze pattern of numbers to be 1 more than the number of rows, so a frieze pattern of order n has $n-1$ rows, the first and last of which are all 1s.

Problem 4 Our additional constraint, satisfied by these examples, is a property of "diamonds" of four numbers. Any time we have numbers a, b, c, d arranged in the following shape, they must satisfy a simple equation. Can you guess what it is? A number frieze with this property is called *unimodular*, and for the rest of the worksheet, we will only care about unimodular number friezes.

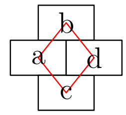

Figure 4: Unimodular condition

2.1 Periodicity in Order 5

Problem 5 Gauss proved the following identity:

$$u_0\left((1+u_4-u_1u_2)-(1+u_1-u_3u_4)\right)=(1+u_2)(1+u_3-u_0u_1)-(1+u_3)(1+u_2-u_4u_0)$$

Use it to show $u_0 = u_5$ in the following frieze pattern, and then describe the rest of the frieze pattern:

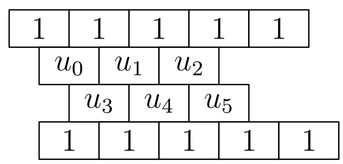

Figure 5

2.2 Constant rows

Let's think for a while about how to build unimodular frieze patterns where each row only consists of one number, repeating.

Problem 6

(a) In a unimodular frieze pattern of order 4, the first and third row are all ones. If the remaining row is also only one number, call it x, repeated, what possible values can x take?

(b) In a unimodular frieze pattern of order 5, there are now two rows to consider between the 1s. Say every entry in the second row is x, and the third row is all y. What can x and y be? How do they have to be related?

Problem 7 Recall this theorem from Euclidean Geometry: If $ABCD$ is a polygon inscribed in a circle, then

$$AB \times CD + BC \times DA = AC \times BD$$

(a) If $ABCDE$ is a regular pentagon with all side lengths 1, can you use that identity to build a frieze pattern where each row consists of only one number, and each such number is the length of a side or diagonal of $ABCDE$?

(b) Can you use this logic to build a frieze pattern of order n for any $n > 5$, still keeping each row constant? How unique is it?

3 Useful Patterns and Formulas

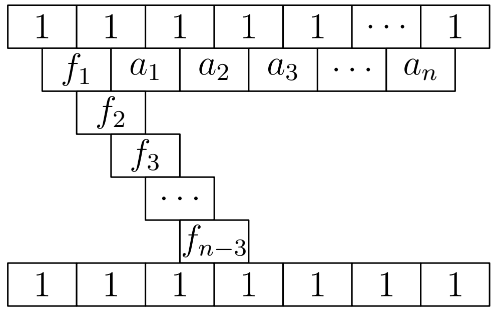

Figure 6: Selected elements of a frieze pattern of order n

Problem 8

(a) Say for now that $n = 8$.

In the above picture, say the diagonal is filled in, that is, $f_1, \ldots, f_5$ are fixed. If we assume this can be extended to *some* frieze pattern, is the rest of the frieze pattern determined uniquely? Can you find formulas for $a_1, \ldots, a_5$ in terms of $f_1, \ldots, f_5$?

(b) In that same picture, say that $a_1, \ldots, a_5$ are fixed. Is the rest of the frieze pattern determined uniquely? Can you find formulas for $f_1, \ldots, f_5$ in terms of $a_1, \ldots, a_5$?

Problem 9 Assume $n \geq 4$. Defining $a_1, \ldots, a_n$ as above, prove that $a_r a_{r-1} > 1$.

Problem 10 We define the *period* of a frieze pattern to be the least positive integer p such that each row repeats every p numbers. In particular, in a pattern of period p, $a_k = a_{k+p}$. The period of a frieze pattern of order n has period p dividing n, and you should assume this for now, but we will not prove it yet.

(a) Determine the period of each frieze pattern in Figure 2.

(b) In Section 2.1, we saw that all frieze patterns of order 5 have either period 5 or period 1. Verify that in the examples from part (a), the period divides the order of the frieze pattern.

Problem 11 Show that for $n = 3, 4$, the period of a frieze pattern of order n is actually strictly less than n. Later, once we know how to construct many frieze patterns of integers, we will show that for $n \geq 5$, there is always at least one frieze pattern (of integers) of order and period exactly n.

Problem 12 As we've shown, the second row of a frieze pattern of order n can be expressed in the form $a_1, a_2, \ldots, a_n, a_1, a_2, \ldots, a_n, \ldots$. Let the following rows be given by $b_1, b_2, \ldots,$ $c_1, c_2, \ldots,$ and so on. If the repeating sequence $a_1, a_2, \ldots, a_n$ is replaced with $1, a_1 + 1, a_2, \ldots, a_{n-1}, a_n + 1$, as in the following diagram, this frieze pattern can be turned into a frieze pattern of order $n + 1$, by inserting new diagonals, as shown in the diagrams below. Solve for the ?s in that frieze pattern, and describe the changes to the overall frieze pattern. We call this new frieze pattern an *expansion* of the original.

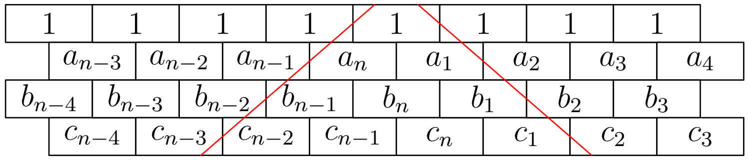

Figure 7: The first 4 rows of the original frieze pattern

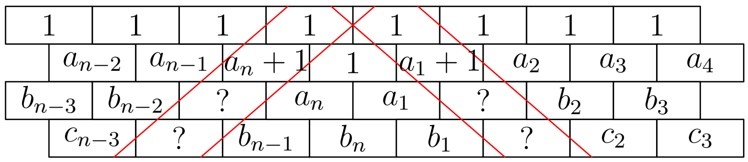

Figure 8: The first 4 rows of the expansion

4 Now with Integers: Quiddity Cycles

We now have several examples of unimodular frieze patterns of numbers, and you may have noticed that several consist entirely of positive natural numbers. We'd like to focus on these now.

We've already observed that in a frieze pattern of order n with only positive numbers, a sequence of n consecutive elements of the second row (immediately below the top row of 1s) determines the whole pattern uniquely. Conway and Coxeter call these numbers, $a_1, \ldots, a_n$, which we know repeat to form the entire second row, a *Quiddity*[1] *cycle* whenever the frieze pattern consists only of positive integers.

Problem 13

(a) Does every quiddity cycle include at least one 1?

(b) Show that every frieze pattern of integers is an expansion of another frieze pattern of integers.

(c) If $a_1, \ldots, a_{n+1}$ is a quiddity cycle coming from the second row of frieze pattern A, and A is the expansion of frieze pattern B which has quiddity cycle $b_1, \ldots, b_n$, solve for $b_1, \ldots, b_n$ in terms of $a_1, \ldots, a_{n+1}$ and vice versa. We call $a_1, \ldots, a_{n+1}$ the expansion of $b_1, \ldots, b_n$.

Problem 14 Starting with the quiddity cycle $1, 1, 1$ of order 3, repeatedly calculate the expansion of that cycle a few times to create some possible quiddity cycles of orders 4, 5, and 6. How many different quiddity cycles can you and the people at your table find? (We consider two quiddity cycles the same if they are mirror images of one another, or if the infinite sequence they make when repeated is the same. For instance, $1, 2, 3, 1, 2, 3$ is basically the same as $2, 3, 1, 2, 3, 1$ or $3, 2, 1, 3, 2, 1$.)

4.1 Triangulated Polygons

A *triangulated regular n-gon* is a regular n-gon which has been partitioned into triangles by drawing $n - 2$ nonintersecting diagonals.

[1]Quiddity means something like "essence," this sequence is the essence of the frieze pattern.

Problem 15

(a) Draw and count all triangulations of an equilateral triangle, a square, a regular pentagon, and a regular hexagon (up to rotations and reflections). Do the numbers of triangulations this bear any similarity to your answers from Problem 14?

(b) Can you find a correspondence between the quiddity cycles of order n and the triangulations of n-gons?

Problem 16 Part 1 of this worksheet contained lots of integer frieze patterns, and their second rows (and second-to-last rows) are all quiddity cycles. What triangulations of polygons do they correspond to?

Problem 17 Using triangulated polygons, verify that for $n \geq 5$, there is a frieze pattern of order and period exactly n.

Problem 18 Show that a frieze pattern of integers has a vertical reflection line if and only if the quiddity cycle in its second row corresponds to a triangulation of a polygon with a reflection symmetry.

5 Challenge Problems

Problem 19 We will now extend the formulas for Problem 8 from that frieze pattern of order 8 to a frieze pattern of order n. For any index i, let g_i be the entry immediately above and to the right of f_i, and use the notation $g_{-1} = -1$ and $f_{-1} = g_0 = 0$, and $f_0 = 1$. Similarly let $f_{n-1} = g_n = 0$ and $f_n = g_{n+1} = -1$. Note that assigning these values would satisfy the unimodular rule.

(a) Define $(r, s) = f_r g_s - f_s g_r$. Prove the following identities:

$$(r, r) = 0$$

$$(r, s) + (s, r) = 0$$

$$(r, s)(t, u) + (r, t)(u, s) + (r, u)(s, t) = 0$$

$$(-1, s) = f_s, (0, s) = g_s$$

(b) Show that the following frieze pattern is unimodular:

	$(0,1)$	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$
$(-1,1)$	$(0,2)$	$(1,3)$	$(2,4)$	$(3,5)$	$(4,6)$
	$(-1,2)$	$(0,3)$	$(1,4)$	$(2,5)$	$(3,6)$
		$(-1,3)$	$(0,4)$	$(1,5)$	$(2,6)$	$(4,7)$
			$\cdots$	$\cdots$	$\cdots$	$\cdots$

(c) Use part (b) to write an equation for a_s in terms of $f_1, \ldots, f_{n-2}$.

(d) Find a recurrence relation for $f_1, \ldots, f_{n-2}$ given $a_1, \ldots, a_n$. If you know the word *determinant*, use it to find a closed-form expression for f_s given $a_1, \ldots, a_n$.

(e: EXTRA CHALLENGE) Using the identities that we have for the expression (r, s), prove that $(r, s) = (r + n, s + n)$, and thus that an order n frieze pattern is periodic with period dividing n.

Problem 20

(a) For which n does there exist a frieze pattern of order n that only contains Fibonacci numbers?

(b) If a frieze pattern of integers does not consist only of Fibonacci numbers, must it contain a 4?

Problem 21 What equation relates $a_0, a_1, \ldots, a_{n-3}$, where $a_0 = f_1$?