GRAPHS

LAMC OLYMPIAD GROUP, WEEK 3

A graph is $G = (V, E)$ where V is a collection of vertices and E is a collection of edges between them. In this handout, we require that there are finitely many vertices, and we do not allow edges from a vertex to itself. Think of vertices as points and of edges as lines connecting them:

- In an ***unoriented graph***, each edge is an unordered pair of vertices (drawn as a straight or curved segment);
- In an ***oriented*** graph, each edge is an ordered pair of vertices (drawn as an arrow).

The ***degree*** $\deg v$ of a vertex v in an unoriented graph G is the number of edges to which it is adjacent. The ***indegree*** $\text{indeg}v$ and ***outdegree*** $\text{outdeg}v$ of a vertex v in an oriented graph G are the number of edges going into, respectively out of v.

A ***path*** in a graph G is a sequence of edges such that each edge ends on the starting vertex of the next edge. A ***cycle*** is a path that begins and ends on the same vertex, *without repeating edges.* A graph is called ***connected*** if there is a path between any two vertices.

A graph is called ***planar*** if it can be drawn on a piece of paper (the mathematical term is that it can be embedded in a plane), where the vertices are points and the edges are straight segments, such that *no two edges intersect.*

Problem 1. Show that the sum of the degrees of all vertices in an unoriented graph $G = (V, E)$ is equal to twice the number of edges. In other words, show that

$$\sum_{v \text{ vertex}} \deg v = 2|E|.$$

Problem 2. Suppose that in an oriented graph $G = (V, E)$, each vertex has indegree and outdegree equal to 1. Prove that G can be broken up into disjoint cycles (that is, G is a union of disjoint graphs, each of which consists only of one cycle).

Bonus: How does this relate to permutations breaking up into cycles?

Problem 3. An unoriented graph is called a *tree* if it is connected and contains no cycles. Show that an unoriented *connected* graph G with n vertices is a tree *if and only if* it is has $n - 1$ edges.

Problem 4. In an ancient city there were n fortresses, and $n - 1$ bridges between them such that any two fortresses could communicate. Princess Ann wants to get from her fortress to the fortress where her sister, Beth, is kept prisoner by an evil wizard. But the wizard cast a spell so that any bridge that Ann crosses collapses immediately afterwards. Show that there is exactly one path that the princess can take.

Problem 5. Let $G = (V, E)$ be a connected unoriented graph such that each cycle has even length. Show that the vertices can be colored with red and blue such that no two neighbors share the same color.

Problem 6. Suppose the maximum degree of any vertex in an unoriented graph $G = (V, E)$ is d. Show that the vertices can be colored with $d+1$ colors such that no two neighbours share the same color.

Problem 7. Let $G = (V, E)$ be an unoriented graph which has no triangles; that is, among any distinct vertices a, b, c, at least two are not neighbors.

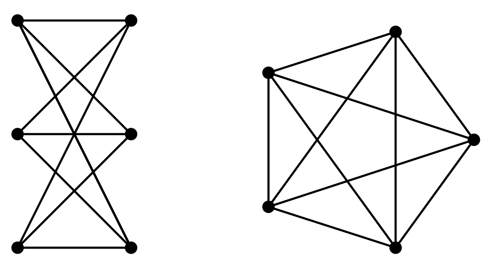
(a) Show that for any pair of neighbors (v, w), there is the inequality $\deg v + \deg w \leq n$, where n is the number of vertices.

(b) Use the previous part to conclude that $\sum_{v \in V} (\deg v)^2 \leq n \cdot e$, where e is the number of edges. *Hint: Consider* $\sum_{\{v,w\} \in E} (\deg v + \deg w)$.

(c) Use this inequality to prove Turàn's Theorem: $e \leq \frac{n^2}{4}$.

(*d) Show that if n is even then there exist a graph for which equality holds: $e = \frac{n^2}{4}$.

Problem 8. Show that the following graphs are not planar (no matter how you draw them):



Problem 9. Let $G = (V, E, F)$ be a connected planar (unoriented) graph, where F is the set of faces (closed regions). Show Euler's relation: $|V| - |E| + |F| = 1$.

Bonus: How does this relate to Problem 3?

Problem 10. Let $G = (V, E, F)$ be a planar (unoriented) graph, where F is the set of faces, such that every vertex has *even* degree. Show that the faces of G can be colored in black and white such that no two adjacent faces (sharing an edge) have the same color.

Problem 11. Suppose the *minimum* degree of any vertex in an unoriented graph $G = (V, E)$ is d. Let $M \subset V$ be a subset of vertices such that any two $v, w \in M$ are at distance at least 3 (the distance is the length of the shortest path between them). Show that

$$|M| \leq \frac{|V|}{d+1}.$$

Problem *12. The *diameter* of a graph is the maximal distance between two vertices. Let $T = (V, E)$ be a tree, and v_0 a vertex. Pick a vertex v_1 such that $d(v_0, v_1)$ is maximal. Then pick v_2 such that $d(v_1, v_2)$ is maximal. Show that $d(v_0, v_2)$ is the diameter.

Remark. This is the quickest method to compute the diameter of a tree.