

Voting Power, Procedures, and Problems

1. Last week, we learned of a voting system in which Cinderella and her stepsisters would vote on which ball to attend. In this system, Cinderella got only 1 vote, while Lucinda and Florinda each got 5 votes, and a majority of votes wins.

- (a) A *winning coalition* is a group which has enough votes to win, if they all vote the same way. For instance, CL is one winning coalition, since Cinderella and Lucinda have a total of 6 votes, which is a majority. There are a total of 4 winning coalitions for the Cinderella-Florinda-Lucinda voting system—list them all!

Solution: CF, CL, FL, CFL

- (b) A member of a winning coalition is *critical* for that coalition if losing that member means the coalition is no longer winning. For instance, in the coalition CL, Cinderella and Lucinda are both critical: if either one of them were removed, the coalition would no longer have enough votes to win. For each winning coalition in part (a), underline its critical members. How many times did you underline each of the voters?

Solution: Cinderella: 2

Florinda: 2

Lucinda: 2

Total number of critical voters (= sum of the 3 numbers above!): 6

2. Here again are the pirate ships of ARRR, along with the number of votes their representative will get (a majority of votes wins):

	Ship	Number of Votes
(A)	Queen Anne's Revenge	9
(B)	Jolly Roger	9
(C)	Black Pearl	7
(D)	The Good Ship Lollipop	3
(E)	Tarantula	1
(F)	Fancy	1

List the possible winning coalitions (you can use the letters A-F), and underline the critical voters in each winning coalition:

Solution: AB, ABD, ABE, ABF, ABDE, ABDF, ABEF, ABDEF,
AC, ACD, ACE, ACF, ACDE, ACDF, ACEF, ACDEF,
BC, BCD, BCE, BCF, BCDE, BCDF, BCEF, BCDEF,
ABC, ABCD, ABCE, ABCF, ABCDE, ABCDF, ABCEF, ABCDEF

3. In the U.N. Security Council, there are 5 permanent members (PMs for short): China, France, Russia, UK, USA; and 10 nonpermanent members (NPMs): currently Austria, Japan, Turkey, Bosnia and Herzegovina, Lebanon, Uganda, Brazil, Mexico, Gabon, and Nigeria. Each PM has veto power over all resolutions, and at least 9 of the 15 members must vote yes for any measure to pass. (Therefore all 5 PMs plus at least 4 NPMs must vote yes for a measure to pass.)

(a) Give a system of weighted votes which gives each member of the Security Council the correct power. For instance, such a system might be "Each PM gets 3 votes, each NPM gets 2 votes, and 20 votes are required to pass a measure"—but this system *doesn't* correctly model the power of each member! Find one that does.

Solution: One possibility is each PM gets 7 votes, each NPM gets 1 vote, and 39 votes are required to pass a measure.

(b) How many winning coalitions are there? (You do not need to list them all.)

Solution: A winning coalition consists of all 5 PMs and *any* set of at least 4 NPMs. Therefore if we count the number of sets of at least 4 NPMs, this will give us the number of winning coalitions. This number is

$$\binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} \\ = 210 + 252 + 210 + 120 + 45 + 10 + 1 = 848$$

(c) For each *permanent* member, on how many winning coalitions is it a critical voter?

Solution: All of them! So, 848.

(d) For each *nonpermanent* member, on how many winning coalitions is it a critical voter?

Solution: An NPM will only be a critical voter if exactly 3 other NPMs are voting yes. There are $\binom{9}{3} = 84$ ways this can happen.

(e) The *power* of a voter is the number of times that voter is critical, divided by the total number of times all voters are critical. What is the power of a permanent member? Of a nonpermanent member?

Solution: The total number of times any voter is critical is $5 * 848 + 10 * 84 = 5080$. So the power of a PM is $848/5080 \approx .167$, and the power of an NPM is $84/5080 \approx .017$. In other words, the permanent members are about 10 times more powerful than the non-permanent members.

4. In the Emerald City Council, the Wizard of Oz gets 5 votes, Dorothy gets 3 votes, and the Tin Man, Scarecrow, and Cowardly Lion get 1 vote each; 8 votes are required to pass a measure.

(a) What is the power of each voter in this system?

Solution: Winning coalitions are WD, WDT, WDS, WDC, WDTS, WDTC, WDSC, WDTSC, WTSC. Therefore the power of each voter is: Wizard 9/19, Dorothy 7/19, Tin Man 1/19, Scarecrow 1/19, Cowardly Lion 1/19.