Lesson 3: Probability II

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Definition 1 (**Conditional Probability**).
Let $A, B \subset \Omega$ be two events, such that $\mathbf{P}(B) \neq 0$. The conditional probability of A given B, written $\mathbf{P}(A|B)$, is the probability of A given that B happened:

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} \tag{1}$$

In particular, if Ω is a finite sample space and every elementary outcome in Ω is equally likely, then

$$\mathbf{P}(A|B) = \frac{|A \cap B|}{|B|} \tag{2}$$

Recall that $|A|$ means the number of elements in A.

Theorem 1 (**Law of Total Probability**).
Suppose that $A, B_1, \ldots, B_n \subset \Omega$, with $B_1, \ldots, B_n$ pairwise disjoint (i.e. $B_i \cap B_j = \varnothing$ if $i \neq j$) and $B_1 \cup \ldots \cup B_n = \Omega$. Then

$$\mathbf{P}(A) = \sum_{i=1}^{n} \mathbf{P}(A|B_i)\mathbf{P}(B_i) \tag{3}$$

1 Introductory problems

Remark 1.
In each problem below, describe how you are using conditional probability and/or the Law of Total Probability.

Problem 1.
a) You roll a 6-sided die and someone (correctly) tells you that the roll was even. What would the probability of getting a 6 be (taking this information into account)?
b) Suppose you roll a 6-sided die twice. Given that the sum of the two numbers is ≥ 8, what is the probability that the sum is 12? Compare this with the probability of getting a sum of 12 with no additional information given.

Problem 2.
Suppose you flip a fair coin 7 times. Given that the first 3 flips are heads, what is the probability of getting exactly 4 heads total?

Problem 3.
Prove the Law of Total Probability.

Problem 4.
Two lines l, k are perpendicular to each other and intersect at O. You roll a 10-sided fair die and pick a point A on l such that $|AO|$ is equal to the number on the die. Then you roll the die again and similarly pick point B on k. What is the probability that the area of $\triangle AOB$ is greater than 20?

2 Advanced problems

Problem 5.
Suppose you have three bags: bag A with 1 red and 3 green balls, bag B with 2R, 2G, and bag C with 1R, 4G. You roll a die and then pick two balls from bag A if you roll $1-3$, B if you roll $4-5$, and C if you roll 6. Find the probability that both balls are the same color.

Problem 6.
Alex wants to celebrate her birthday with friends. Dad offers her a deal: "There are 3 days until your birthday. In each of these days, you play a chess game with me or mom, alternating opponents, one game per day. You can choose who to play with first. If you win 2 games in a row – you can celebrate with friends." Who should Alex play first, if dad plays better than mom (i.e the probability of winning against dad is less)?

Problem 7 (Bayes' Rule).
Let $A, B \subset \Omega$ be events such that $\mathbf{P}(B) > 0$. Then

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(B|A)\mathbf{P}(A)}{\mathbf{P}(B)} \tag{4}$$

Problem 8.
A test for a certain rare disease is assumed to be correct 95% of the time: if a person has the disease, the test results are positive with probability 0.95, and if the person does not have the disease, the test results are negative with probability 0.95. The disease affects 1 out of every 1000 people in the population randomly. Given that the person just tested positive, what is the probability of actually having the disease?