DIGITS AND BASE REPRESENTATIONS
PROBLEM SOLUTIONS

LAMC OLYMPIAD GROUP, WEEK 1

Problem 1. **(a)** Note that any two numbers that are congruent modulo 9 are also congruent modulo 3. So it suffices to only give a proof for 9. Indeed, we must prove that for every number $A_1A_2\dots A_n$ in base 10, we have

$$A_1A_2\cdots A_n \equiv A_1 + A_2 + \cdots + A_n (\text{mod } 9)$$

The idea is to observe that $10 \equiv 1 (\text{mod } 9)$ implies $10^k \equiv 1 (\text{mod } 9)$ for all $k \geq 1$. When we write the number in expanded form, all powers of 10 can be replaced by 1:

$$A_1A_2\cdots A_n = A_1 \times 10^{n-1} + A_2 \times 10^{n-2} + \cdots + A_n \equiv A_1 + A_2 + \cdots + A_n (\text{mod } 9).$$

An alternative solution not using modulos comes from the observation that $10^k - 1 = 99\cdots 9$ is divisible by nine. Thus,

$$A_1A_2\cdots A_n - (A_1 + A_2 + \cdots + A_n) = A_{n-1} \times 9 + A_{n-2} \times 99 + \cdots + A_1 \times 99\cdots 9,$$

which is divisible by nine.

(b) We apply the same idea, only this time $10 \equiv -1 (\text{mod } 11)$ implies $10^k \equiv (-1)^k (\text{mod } 11)$ for all $k \geq 1$. So when we expand, we get the desired result

$$\begin{aligned} A_1A_2\cdots A_n &= A_1 \times 10^{n-1} + A_2 \times 10n - 2 + \cdots + A_n \\ &\equiv A_1 \times (-1)^{n-1} + A_2 \times (-1)^{n-2} + \cdots - A_2 + A_1 (\text{mod } 11). \end{aligned}$$

(c) Note that 2^a5^b divides $2^{\max(a,b)}5^{\max(a,b)} = 10^{\max(a,b)}$. Hence, any two numbers congruent modulo $10^{\max(a,b)}$ are also congruent modulo 2^a5^b. Since the residue class modulo $10^{\max(a,b)}$ is given by the last $\max(a,b)$ digits, we learn that the last $\max(a,b)$ digits also determine the residue class module 2^a5^b. □

Problem 2. The number of zeros in $n!$ is the minimum between the exponent of 2 in $n!$ (call this a), and the exponent of 5 in $n!$ (call this b). Indeed, if

$$n! = 2^a \cdot 5^b \cdot m$$

where m is relatively prime (coprime) with 10, then the maximal power of 10 dividing $n!$ is $10^{\min(a,b)}$. So it remains to compute a and b, and figure out which of them is smaller.

To compute a (the exponent of 2), note that each even number between 1 and n (that is, $2, 4, 6, \dots, 2\lfloor n/2 \rfloor$) contributes at least one factor of 2 to the product $n! = 1 \cdot 2 \cdots (n-1) \cdot n$. Overall, they contribute $\lfloor n/2 \rfloor$ units to the exponent a. But the multiples of 4 (that is, $4, 8, 12, \dots, 4\lfloor n/4 \rfloor$) each contribute an extra factor of 2, so to account for this we have to add $\lfloor n/4 \rfloor$ to a. We continue this counting process for 8, 16, 32, etc., until we've eventually used all the powers of 2 that are less than or equal to n. Overall, we have that

$$a = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{2^k} \right\rfloor,$$

where 2^k is the largest power of 2 that is less than or equal to n. By the exact same argument,

$$b = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{5^q} \right\rfloor,$$

where 5^q is the largest power of 5 that is less than or equal to n. Note that it is okay to include larger powers of 5 in this count, since for $j > q$, we have $5^j > n$ so $\lfloor n/5^j \rfloor = 0$. Hence we may as well write

$$\begin{aligned} b &= \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{5^{\max(k,q)}} \right\rfloor \\ &\leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{2^{\max(k,q)}} \right\rfloor = a, \end{aligned}$$

since $2 < 5 \Rightarrow \frac{n}{2} > \frac{n}{5} \Rightarrow \left\lfloor \frac{n}{2} \right\rfloor \geq \left\lfloor \frac{n}{5} \right\rfloor$, and so on. (In fact, it is easy to see that $k \geq q$, so $\max(k, q) = k$). Hence $\min(a, b) = b$; in other words, in forming zeros at the end of $n!$, we run out of 5's before we run out of 2's. The number of zeros at the end of $n!$ is thus $b = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{5^q} \right\rfloor$. □

Remark. We implicitly used the fact that 2 and 5 are prime, so this type of argument ***wouldn't work*** to show ***the wrong statement*** that the exponent of 10 in $n!$ is

$$\left\lfloor \frac{n}{10} \right\rfloor + \left\lfloor \frac{n}{100} \right\rfloor + \left\lfloor \frac{n}{1000} \right\rfloor + \cdots$$

This is because a 10 can appear from the product of a number divisible by 2 and one divisible by 5 (rather than only from numbers that are directly divisible by 10); for primes we don't have this problem, since they are indivisible.

So, if we worked in a general base $B \geq 1$ *instead of base* 10, *think about how you could use the prime factorization of* B *to answer in how many zeros* $(n!)_B$ *ends.*

Problem 3. By Problem 1 (a), any positive integer n is congruent modulo 9 to the sum of its digits, which we denote by $S(n)$. But using Problem 1 (a) again, $S(n)$ is also congruent modulo 9 to the sum of its digits, $S(S(n))$. Repeating this process sufficiently many times, we get

$$n \equiv S(n) \equiv S(S(n)) \equiv \cdots \equiv \text{Superdigit}(n) \pmod 9$$

In other words, the residue (or congruence class) modulo 9 *is an* **invariant** *under the operation of taking the sum of digits, so no matter how many times we apply this operation, this invariant stays the same. Remember that invariants are very important in Combinatorics, but here we see that they also show up in Number Theory.*

But 5 is the only digit congruent to 5 modulo 9, so a positive integer has superdigit 5 if and only if it is congruent to 5 modulo 9. Up to 2019, the positive integers congruent to 5 modulo 9 are $9 \cdot 0 + 5, 9 \cdot 1 + 5, \ldots, 9 \cdot 223 + 5$, so the answer is $223 - 0 + 1 = 224$. □

Remark. The term "superdigit" isn't standard, so don't use it outside of this problem. Also, you may wonder if the superdigit of a number is well-defined, that is, if for every positive integer n we eventually reach a single digit by repeatedly taking the sum of digits: $S(n), S(S(n)), \cdots$. This is, indeed, not hard to prove: try showing that if n has at least two digits, then $S(n) < n$. Since $S(n)$ is always positive, we cannot go down indefinitely in the tower of inequalities $n > S(n) > S(S(n)) > \cdots$. Hence at some point in this tower we must reach a number d such that $S(d) = d$, that is d is a digit (the superdigit).

Problem 4. Let us denote the number ABCDE by the letter x. Then the condition $3 \times$ 2ABCDE = ABCDE2 becomes

$$3 \cdot (200000 + x) = 10 \cdot x + 2.$$

Now, we can simply solve for x:

$$600000 + 3x = 10x + 2 \iff 7x = 599998 \iff x = \frac{599998}{7} = 85714.$$

Hence, $\mathrm{A} = 8, \mathrm{B} = 5, \mathrm{C} = 7, \mathrm{D} = 1, \mathrm{E} = 4$. Note that another way to arrive at the answer is to deduce the letters one by one, starting from $E, D, ...$ and ending with A. However, this method is simpler and more elegant.

Problem 5. Let us first assume that 2 and 5 do not divide n, and prove that $\frac{m}{n}$ can be represented decimally without a non-periodic part. To this end, we recall the celebrated theorem of Euler, which states that

$$a^{\varphi(n)} \equiv 1 (\text{mod } n),$$

for all integers a coprime to n.[1] We can apply this for $a = 10$, because we have assumed that 2 and 5 do not divide n. This means that n divides $10^{\varphi(n)} - 1$, that is that $10^{\varphi(n)} - 1 = nd$ for some positive integer d. Thus,

$$\frac{m}{n} = \frac{md}{nd} = \frac{md}{10^{\varphi(n)} - 1}.$$

We extract the greatest multiple of $10^{\varphi}(n) - 1$ from the numerator md:

$$md = q(10^{\varphi(n)} - 1) + r,$$

and we are left with

$$\frac{m}{n} = q + \frac{r}{10^{\varphi(n)} - 1}.$$

where q is an integer, and $0 \leq r < 10^{\varphi(n) - 1}$. The term $\frac{r}{10^{\varphi(n)} - 1}$ has no nonperiodic part, due to the formula for $0.\overline{C_1 C_2 \cdots C_p}$ from class. This proves the claim.

Assuming now that $\frac{m}{n}$ has no nonperiodic part, we have

$$\frac{m}{n} = A_1 \cdots A_m . \overline{C_1 \cdots C_p} = A_1 \cdots A_m + \frac{C_1 \cdots C_p}{10^p - 1}.$$

The denominator of this fraction in reduced form is a divisor of $10^p - 1$, which indeed is not divisible by 2 or 5. $\square$

Problem 6. Note that the number of digits of 2^n (or any positive integer) is determined by the smallest power of 10 bigger than it. Hence we basically want to compare (and in fact, approximate) powers of 2 with powers of 10 in a somewhat efficient way. To do this, it is natural to search for powers of 2 that are close to powers of 10; we find that

$$2^3 = 8 < 10, \qquad 2^{10} = 1024 > 10^3.$$

(Remember that $2^{10} = 1024$ is very close to 1000; this shows up a lot, especially in Computer Science). We then try to take as many 1024's out of 2^n as we can, resulting in

$$\begin{aligned} 2^n = 2^{10(n/10)} &= 1024^{n/10} \\ &\geq 1000^{n/10} = 10^{0.3 \cdot n} \geq 10^{\lfloor 0.3 \cdot n \rfloor}, \end{aligned}$$

[1]Here, $\varphi(n)$ is used to denote the number of integers between 1 and n that are coprime to n.

where the last number has exactly $\lfloor 0.3 \cdot n \rfloor + 1 \geq 0.3 \cdot n$ digits (note that $0.3 \cdot n$ is not in general an integer, but it still makes sense to compare it with integers).

Recall that for any real number x, the **largest integer less than or equal to** x is denoted $\lfloor x \rfloor$, and called the **floor** of x. Symmetrically, the **smallest integer greater than or equal to** x is denoted $\lceil x \rceil$, and called the **ceiling** of x. Please don't use any other notations.

Similarly, for the upper bound, assume WLOG that $n \geq 1$ (at $n = 0$ the claim is trivial). Then we can write

$$2^n = 2^{3(n/3)} = 8^{n/3}$$
$$< 10^{n/3} \leq 10^{\lceil 0.\overline{3} \cdot n \rceil}$$

Since 2^n is strictly less than the above power of 10, we find that it has at most $\lceil 0.\overline{3} \cdot n \rceil \leq 0.\overline{3} \cdot n + 1$ digits. $\square$

Problem 7. Recall that a strictly positive integer n has k digits if and only if the inequality $10^{k-1} \leq n < 10^k$ holds. If we denote by a and b the number of digits in 2^n and 5^n respectively, then we have the two inequalities

$$10^{a-1} \leq 2^n < 10^a$$
$$10^{b-1} \leq 5^n < 10^b.$$

Furthermore, notice that the equality case in the left side cannot occur, since 2^n and 5^n are not powers of ten. Since everything is positive, we can multiply the two inequalities (which are now strict), to get

$$10^{a+b-2} < 10^n < 10^{a+b}.$$

Since powers of ten increase strictly when the exponent increases, we can derive that $a+b-2 < n < a+b$. Since a, b, n are integers, the only possibility left is $n = a+b-1$, or equivalently $a + b = n + 1$. This shows that the total number of digits in 2^n and 5^n for $n > 0$ is equal to $n + 1$.

Problems 8 and 9. These were given as homework, but they're not very hard. Try starting with smaller cases ($n = 0, 1, 2$) and prove your statement in that case (for 8), or guess a pattern (for 9). Recall that in class we solved Problem 9(a) by showing that

$$\underbrace{99\cdots9}_{n\text{ digits}}8\underbrace{00\cdots0}_{n\text{ digits}}1 = (\underbrace{99\cdots9}_{n+1\text{ digits}})^2.$$

Problem 10. The trick here is to notice that $10^{2n} - 1 = (10^n - 1)(10^n + 1)$. Indeed, we can use this to deduce

$$\frac{1}{10^n + 1} = \frac{10^n - 1}{10^{2n} - 1} = \frac{00\cdots0 99\cdots9}{99\cdots9 99\cdots9},$$

where both the blue and the red part each have n digits. Hence,

$$\frac{1}{10^n+1} = 0.\overline{00\cdots0 99\cdots9} = 0.00\cdots0 99\cdots9 00\cdots0 99\cdots9 00\cdots0 99\cdots9\cdots$$

is the decimal expansion.

We'll let you work on Problems 11, 12 and 13 from the handout on digits and base representations for as long as you want, and you can turn in any solution to them. In particular, if you solve 11 or 12, you'll get a lot of bonus points in our ranking.