LAMC Week 1: Induction and Bezout's Lemma

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1 Problems

1. Find all natural numbers k such that $2^k \leq k^2$.

2. Prove that, except $x = y = z = 0$, there is no other triple $(x, y, z) \in \mathbb{Z}^3$ such that $x^2 + y^2 = 3z^2$.

3. (Binomial theorem) Prove that for all natural numbers n the following holds:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

4. (Fermat's little theorem) Show that if p is a prime number, then for all natural numbers a, p divides $a^p - a$.

5. (The very first IMO problem, 1959 #1) Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

6. (ACM-ICPC Qualifier 2019 (a programming contest)) A rectangular chocolate bar forms an $m \times n$ grid of squares, with m, n positive integers. If we cut the chocolate bar along the diagonal, how many squares are divided into two parts of equal size?

7. (Putnam 2000) For $m \leq n$ positive integers, show that

$$\frac{\gcd(m,n)}{n} \binom{n}{m}$$

   is an integer.

8. (Chinese remainder theorem) Let $a_1, a_2, \ldots a_n$ be pairwise relatively prime positive integers, and let $b_1, b_2, \ldots b_n$ be any integers. Show that there exists an integer B such that $B \equiv b_i \mod a_i$ for all i from 1 to n.

9. (Bonus) Let $\{a_i\}$ be a sequence of real numbers that satisfy

$$a_{i+j} \leq a_i + a_j \text{ , } \forall\, i,\, j \in \mathbb{N}$$

   Prove that

$$\frac{a_1}{1} + ... + \frac{a_n}{n} \geq a_n$$