LENGTHS AND AREAS

LAMC OLYMPIAD GROUP, WEEK 2

Very often, geometry problems ask you to compute a certain quantity in a figure (a length, an area, an angle), given some initial data. You'll only master this type of problems after you learn a bit of trigonometry, but today we'll avoid using the words sin, cos, tan.

Throughout, for any triangle $\triangle ABC$ we denote $a = BC$, $b = CA$, $c = AB$ for simplicity (these are relatively standard notations). Also, the area of a triangle $\triangle ABC$ will be denoted $[ABC]$, and the area of a quadrilateral $ABCD$ will be denoted $[ABCD]$. As a bare minimum, it's important that you know the facts below.

(1) The area of a triangle $\triangle ABC$ is $\frac{ah_A}{2}$, where h_A is the length of the height from A.
(2) The area of an *equilateral* triangle $\triangle ABC$ of side ℓ is $\frac{\ell^2\sqrt{3}}{4}$.
(3) In a right triangle $\triangle ABC$ with $\angle A = 90°$ and $\angle B = 30°$, one has $a = 2b$.
(4) (*Pythagoras*) In a right triangle $\triangle ABC$ with $\angle A = 90°$, one has $a^2 = b^2 + c^2$.
(5) In two similar triangles $\triangle ABC \sim \triangle A'B'C'$, one has $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$.
(6) The area of a circle of radius r is πr^2, and the circumference is $2\pi r$. The area of a sector of a circle of angle α is proportional to the angle, thus equal to $\frac{\alpha}{360}\pi r^2$.

Problem 1. In a parallelogram $ABCD$, let $\ell := AB = CD$, and d be the distance between AB and CD. Compute the area $[ABCD]$.

Problem 2. Let $\triangle ABC$ be a right triangle, with $\angle A = 90°$. Let D be the foot of the perpendicular on side BC from vertex A.
(a) If $AB = 3$ and $AC = 4$, find all other lengths in the picture (BD, AD, CD)
(b) If $CD = 4$ and $BD = 9$, find all other lengths in the picture (AB, AD, AC)
(c) If $AD = 5$ and $AC = 13$, find all other lengths in the picture (AB, BD, DC).

Problem 3. Let $ABCD$ be a convex quadrilateral such that $AB \perp AD$, $AD \perp DC$ and $AC \perp BD$. If $AB = 4$ and $CD = 9$, what are the lengths of AD and BC?

Problem 4. (a) Let $\triangle ABC$ be a triangle with $\angle A = 60°$. Prove that $a^2 = b^2 + c^2 - bc$.
(b) Let $\triangle ABC$ be a triangle with $\angle A = 120°$. Prove that $a^2 = b^2 + c^2 + bc$.

Problem 5. In a rectangle $ABCD$, let M, N, P, Q be the midpoints of AB, BC, CD, DA respectively. Let X be a point inside $ABCD$. Show that

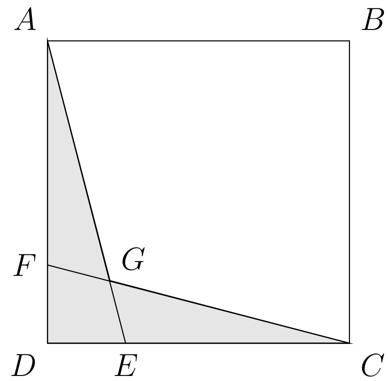
$$[AMXQ] + [CPXN] = [BNXM] + [DQXP]$$

Problem 6. Let $ABCD$ be a convex quadrilateral with given side lengths $AB = 3, BC = 13, CD = 12, DA = 4, BD = 5$. Find out the length of AC.

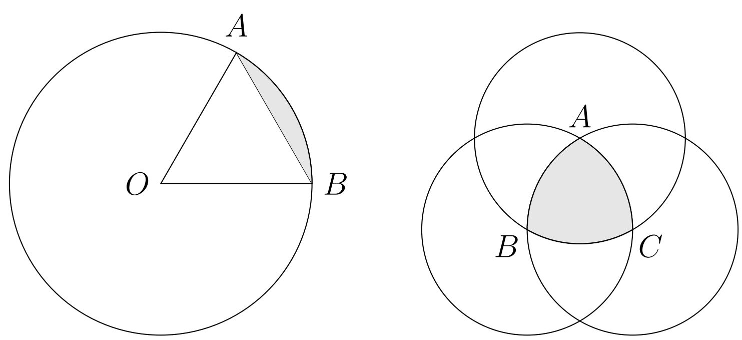
Problem 7. In a square $ABCD$ of side length 1, let $E \in \overline{CD}$ and $F \in \overline{AD}$ such that $\angle DAE = \angle DCF = 15°$.

(a) Compute the area of the shaded region below.

(b) Given $x := DE$ (which you don't need to compute) and $G := CF \cap AE$, compute the area of the quadrilateral $DEGF$.



Problem 8. **(a)** Let O be a point and $\mathscr{C}$ be a circle centered at O of radius r. Take two points A and B on the circle such that $\angle AOB = 60°$. What is the area of the shaded region (left picture)?



(b) Let A, B, C form an equilateral triangle of side length 1, and draw circles of radius 1 around A, B and C. Compute the area of the shaded region.