DIGITS AND BASE REPRESENTATIONS

LAMC OLYMPIAD GROUP, WEEK 1

Let $\mathrm{A}_1, \dots, \mathrm{A}_m$, $\mathrm{B}_1, \dots, \mathrm{B}_n$, $\mathrm{C}_1, \dots, \mathrm{C}_p$ be (**not necessarily distinct**) digits in base 10. Then we have

$$\mathrm{A}_1 \cdots \mathrm{A}_m \,.\, \mathrm{B}_1 \cdots \mathrm{B}_n \overline{\mathrm{C}_1 \cdots \mathrm{C}_p} = \mathrm{A}_1 \cdots \mathrm{A}_m + \frac{\mathrm{B}_1 \cdots \mathrm{B}_n}{10^n} + \frac{\mathrm{C}_1 \cdots \mathrm{C}_p}{10^n(10^p - 1)}.$$

Above, p is called the period. For example, one has

$$0.\overline{142857} = 0.14285714285714 \cdots = \frac{142857}{999999} = \frac{1}{7}.$$

Of course, there's nothing special about base 10, mathematically speaking. In general, we can write a number in base $b \geq 1$ (with digits in $\{0, 1, \dots, b-1\}$) as

$$\left(\mathrm{A}_1 \cdots \mathrm{A}_m \,.\, \mathrm{B}_1 \cdots \mathrm{B}_n \overline{\mathrm{C}_1 \cdots \mathrm{C}_p}\right)_b = (\mathrm{A}_1 \cdots \mathrm{A}_m)_b + \frac{(\mathrm{B}_1 \cdots \mathrm{B}_n)_b}{b^n} + \frac{(\mathrm{C}_1 \cdots \mathrm{C}_p)_b}{b^n(b^p - 1)},$$

where

$$(A_1 \cdots A_m)_b = A_m + A_{m-1}b + A_{m-2}b^2 + \cdots + A_1 b^{m-1}.$$

Problem 1. Show that in base 10, the following are true (a, b denote nonnegative integers):
(a) A natural number is congruent modulo 3 and 9 to the sum of its digits;
(b) A natural number $\mathrm{A}_n \cdots \mathrm{A}_0$ is congruent modulo 11 to the alternating sum of its digits, $\mathrm{A}_0 - \mathrm{A}_1 + \mathrm{A}_2 - \cdots + (-1)^n \mathrm{A}_n$;
(c) A natural number's residue modulo $2^a 5^b$ only depends on its last $\max(a, b)$ digits.
In particular, these give divisibility criteria (when the residue modulo $3, 9, 11, 2, 5$, etc. is 0).

Problem 2. For integer $n \geq 1$, with how many digits of 0 does $n! = 1 \cdot 2 \cdots (n-1) \cdot n$ end? (Find a formula that is as simple as possible).

Problem 3. Define the *superdigit* of a positive integer n as follows: compute the sum of digits of n; if this sum is ≥ 10, compute the sum of digits of the sum, and repeat this process until the result is a digit. How many numbers in $\{1, \dots, 2019\}$ have the superdigit 5?

Problem 4. Find all choices of digits $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$ in base 10 such that

$$3 \cdot 2\mathrm{ABCDE} = \mathrm{ABCDE}2.$$

Problem 5. Given relatively prime positive integers m and n, show that the rational number $\frac{m}{n}$ can be represented in base 10 *without a nonperiodic part* after the dot if and only if n is not divisible by 2 or 5. *How does this generalize to any base b?*

Problem 6. Show that for $n \geq 0$, the number 2^n has at least $0.3 \cdot n$ digits and at most $0.\overline{3} \cdot n + 1$ digits.

Problem 7. Show that the number of digits in 2^n plus the number of digits in 5^n always equals $n + 1$, for $n > 0$.

Problem 8. Show that for any nonnegative integer n,

$$1\underbrace{33\cdots 3}_{n \text{ digits}}25 \cdot 25 = \underbrace{33\cdots 3}_{n+1 \text{ digits}}125$$

Problem 9. **(a)** Show that

$$\underbrace{99\cdots 9}_{n \text{ digits}}8\underbrace{00\cdots 0}_{n \text{ digits}}1$$

is a perfect square for all integers $n \geq 0$.
(b) Show that

$$\underbrace{99\cdots 9}_{n \text{ digits}}4\underbrace{00\cdots 0}_{n \text{ digits}}9$$

is a perfect square for all integers $n \geq 0$.
(*c) Show that

$$\underbrace{99\cdots 9}_{n \text{ digits}}7\underbrace{00\cdots 0}_{n \text{ digits}}2\underbrace{99\cdots 9}_{n+1 \text{ digits}}$$

is a perfect cube for all integers $n \geq 0$.

Problem 10. Find the decimal expansion of $\frac{1}{10^n+1}$ for all positive integers n.

Problem 11. Let $p \notin \{2, 5\}$ be a prime such that $1, 10, 10^2, 10^3, \cdots$ cover all residues modulo p. Show that the periodic parts in base 10 of $\frac{1}{p}, \frac{2}{p}, \cdots, \frac{p-1}{p}$ are all permutations of each other. For example, if $p = 7$,

$$\frac{1}{7} = 0.\overline{142857}, \quad \frac{2}{7} = 0.\overline{285714}, \quad \frac{3}{7} = 0.\overline{428571},$$
$$\frac{4}{7} = 0.\overline{571428}, \quad \frac{5}{7} = 0.\overline{714285}, \quad \frac{6}{7} = 0.\overline{857142}.$$

How does this generalize to any base b?

Problem *12. Show that for any sequence of digits $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n$ with $\mathrm{A}_1 \neq 0$, there exists a perfect square starting with those digits. That is, show that

$$x^2 = \mathrm{A}_1\mathrm{A}_2\cdots\mathrm{A}_n\mathrm{B}_1\mathrm{B}_2\cdots\mathrm{B}_m.$$

for some integer x.

Problem *13. Solve the equation

$$2^n + 1 = \mathrm{AABBCCDD},$$

for positive integers n and digits A, B, C, D (A $\neq 0$).