Egyptian Multiplication

Beginners Circle 10/15/2017

Ancient Egyptians had an interesting method for multiplying two numbers. Suppose that you have to multiply two numbers (e.g., 23 and 18). The basic operation for them was multiplying a number by 2. They reduced all other multiplication problems to it. Here is how they would start multiplying 23 by 18 (in modern notation):

	23	18	$\circledast$	*Rows used*
	--	--	--	--
	1	**18**	$= 1 \cdot 18$	✓
	2	**36**	$= 2 \cdot 18$	✓
$23 \cdot 18$:	**4**	72	$= 4 \cdot 18$	✓
	8	144	$= 8 \cdot 18$	
	16	**288**	$= 16 \cdot 18$	✓

Here is what they did to complete the multiplication

1. Below the first number (in this case, 23), they would write all of the powers of 2 that are smaller or equal to the number. In the example above, these powers of 2 are 1, 2, 4, 8, and 16.
2. In the second column, they would keep doubling the second number (in this case, 18). This produces the list 18, 36, 72, 144, and 288.
3. After that, they would represent the first number as the sum of the powers of 2 (so that each of the powers of 2 is used at most once).
   For example, if the first number is 23, they would find
   $$23 = 16 + 4 + 2 + 1.$$
   After that, they would mark those rows where these powers of 2 are present in the left column. (In our example, the first, the second, the third, and the fifth rows are marked).

Finally, all there is to do at this point is to add the marked numbers in the second column:

$$\begin{array}{cccc} & & 1 & 8 \\ & & 3 & 6 \\ & & 7 & 2 \\ + & 2 & 8 & 8 \\ \hline & 4 & 1 & 4 \end{array}$$

Thus, the result of the multiplication is 414.

In modern notation, we can rewrite the Egyptian Multiplication algorithm in the following way:

$$23 \cdot 18 = (1 + 2 + 4 + 16) \cdot 18 = 1 \cdot 18 + 2 \cdot 18 + 4 \cdot 18 + 16 \cdot 18$$

1. We will now use Egyptian Multiplication to multiply 13 by 22. Write all powers of two less than 13 in the first column and double each row going down to fill the right-side column.

13·22:

13	22		Rows Used
1	22	= 1·22	✓
2	44	= 2·22	
4	88	= 4·22	✓
8	176	= 8·22	✓

(b) Represent 13 as a sum of powers of 2.

13 = 8+4+1

(c) Finish the Egyptian Multiplication to find 13 times 22.

$$\begin{array}{rl} 22 & (1\cdot 22) \\ 88 & (4\cdot 22) \\ +176 & (8\cdot 22) \\ \hline 286 & \end{array}$$

(d) When performing your Egyptian Multiplication did you start with the highest or lowest power of 2?

We start with the highest power of two to ensure that the representation of 13 as a sum of powers of 2 has the least # of terms.

2. Explain how each number in the second column is obtained from the number in the first column.

E.g. How do you get

- 88 from 4 and 22?

22 goes into 88 four times. Thus, when we multiply 22 by 4, we obtain 88.

- 176 from 8 and 22?

22 goes into 176 eight times. Thus, when we multiply 22 by eight, we obtain 176.

3. Using what you noticed in question 2 do the following:

(a) Rewrite each term in the sum: $18 + 36 + 72 + 144$ as a product of 18 and a power of 2. For example,

$$144 = 8 \cdot 18$$

- $18 = 18 \cdot 2^0 = 18 \cdot 1$
- $36 = 18 \cdot 2^1 = 18 \cdot 1$
- $72 = 18 \cdot 2^2 = 18 \cdot 4$
- $288 = 18 \cdot 2^4 = 18 \cdot 16$

(b) Finish the expression on the right side:

- $18 + 36 + 72 + 288 = (18 \cdot 1) + (18 \cdot 2) + \overbrace{(18 \cdot 4)}^{=72} + \overbrace{(18 \cdot 16)}^{=288}$

(c) What do you notice? Can you simplify this expression by factoring out 18?

I notice that 18 is a common factor. Therefore, I can factor out 18. Thus simplifying it.

$18 \cdot (1 + 2 + 4 + 16)$

4. Multiply the following numbers using Egyptian Multiplication:

(a) 13×41

$13 = 8 + 4 + 1$

13	41		Row Used
1	41	$= 1 \cdot 41$	✓
2	82	$= 2 \cdot 41$	
4	164	$= 4 \cdot 41$	✓
8	328	$= 8 \cdot 41$	✓

$$\begin{array}{rl} 41 & (1 \cdot 41) \\ 164 & (4 \cdot 41) \\ +\ 328 & (8 \cdot 41) \\ \hline 533 & \end{array}$$

(b) 41×13

$41 = 32 + 8 + 1$

41	13		Row Used
1	13	$= 1 \cdot 13$	✓
2	26	$= 2 \cdot 13$	
4	52	$= 4 \cdot 13$	
8	104	$= 8 \cdot 13$	✓
16	208	$= 16 \cdot 13$	
32	416	$= 32 \cdot 13$	✓

$$\begin{array}{r} 13 \\ 104 \\ +\ 416 \\ \hline 533 \end{array}$$

1. Given two numbers, which one (smaller or larger) will you use as the first number in Egyptian Multiplication? Why? Give an example to justify your answer.

Example 1: 17×38

17	38		Row Used
1	38	$= 1 \cdot 38$	✓
2	76	$= 2 \cdot 38$	
4	152	$= 4 \cdot 38$	
8	304	$= 8 \cdot 38$	
16	608	$= 16 \cdot 38$	✓

$17 = 16 + 1$

$$\begin{array}{rr} = & 608 \\ + & 38 \\ \hline & 646 \end{array}$$

Example 2: 38×17

38	17		Row Used
1	17	$= 1 \cdot 17$	
2	34	$= 2 \cdot 17$	✓
4	68	$= 4 \cdot 17$	✓
8	136	$= 8 \cdot 17$	
16	272	$= 16 \cdot 17$	
32	544	$= 32 \cdot 17$	✓

$38 = 32 + 4 + 2$

$$\begin{array}{r} 544 \\ +\ 68 \\ +\ 34 \\ \hline 646 \end{array}$$

I would use the smaller number first because:

- We can write fewer powers of 2 that are smaller or equal to the first number in the first column.
- The number of terms generated when breaking the first number down as a sum of the powers of two would be less.
  - This leads to fewer rows used and thus fewer terms to add.

Notice: if the two numbers are the same, then the order at which the # is used does not matter.

2. Explain in your own words how Egyptian Multiplication works.

Steps: 1) Create two columns
2) List out all the powers of two less than or equal to the smallest of the two number.
3) keep doubling the 2nd # in the second column
4) Represent the first number as the sum of the powers of 2 so that each of the powers are used at most once.
5.) Mark the Rows where the powers of 2 are present in the first coloumn.
6.) Add the numbers in the second column of the marked rows.

Given two numbers, Egyptian Multiplication works by breaking down one of the two numbers into the sum of the powers of 2, multiplying each of the terms of the sum by the second number, and finally summing the results of the multiplication to obtain the answer.

3. With a partner, have a race to see who can multiply numbers faster. One of you must use Egyptian Multiplication and the other must use regular, long multiplication. Race 6 times alternating the type of multiplication you do. Show your work below:

(a) 25 × 31

$25 = 16 + 8 + 1$

$$\begin{array}{r} 31 \\ 248 \\ +\ 496 \\ \hline 775 \end{array}$$

EM:

25	31		Rows Used
1	31	= 31·1	✓
2	62	= 32·2	
4	124	= 32·4	
8	248	= 31·8	✓
16	496	= 31·16	✓

RM:

$$\begin{array}{r} 25 \\ \times 31 \\ \hline 25 \\ +\ 750 \\ \hline 775 \end{array}$$

(b) 38 × 45

$38 = 32 + 4 + 2$

$$\begin{array}{r} 90 \\ 180 \\ +1440 \\ \hline 1710 \end{array}$$

38	45	Rows Used
1	45	
2	90	✓
4	180	✓
8	360	
16	720	
32	1440	✓

$$\begin{array}{r} 38 \\ \times 45 \\ \hline 190 \\ +1520 \\ \hline 1710 \end{array}$$

(c) 12 × 63

$12 = 8 + 4$

$$\begin{array}{r} 504 \\ +\ 252 \\ \hline 756 \end{array}$$

12	63	Rows Used
1	63	
2	126	
4	252	✓
8	504	✓

$$\begin{array}{r} 12 \\ \times 63 \\ \hline 36 \\ +720 \\ \hline 756 \end{array}$$

(d) 17×52

$17 = 16 + 1$

$$\begin{array}{r} 832 \\ +\ 52 \\ \hline 884 \end{array}$$

17	52	Row Used
1	52	✓
2	104	
4	208	
8	416	
16	832	✓

$$\begin{array}{r} 17 \\ \times 52 \\ \hline 34 \\ +850 \\ \hline 884 \end{array}$$

(e) 112×85

$85 = 64 + 16 + 4 + 1$

$$\begin{array}{r} 7168 \\ 1792 \\ 448 \\ +\ 112 \\ \hline 9520 \end{array}$$

85	112	Rows Used
1	112	✓
2	224	
4	448	✓
8	896	
16	1792	✓
32	~~3744~~ 3584	
64	7168	✓

$$\begin{array}{r} 112 \\ \times 85 \\ \hline 560 \\ +8960 \\ \hline 9520 \end{array}$$

(f) 256×50

$50 = 32 + 16 + 2$

$$\begin{array}{r} 8192 \\ 4096 \\ +\ 512 \\ \hline 12{,}800 \end{array}$$

50	256	Row Used
1	256	
2	512	✓
4	1024	
8	2048	
16	4096	✓
32	8192	✓

$$\begin{array}{r} 256 \\ \times 50 \\ \hline 12{,}800 \end{array}$$