

How To Divide Up Seats Among Pirates, and More

1. Liz and Daniel want to split a cake between them (they are very hungry). But they want to make sure that each person gets at least half the cake, *from that person's own perspective*. So Liz cuts the cake into two pieces of equal size (from her perspective), and Daniel chooses the one which is bigger (from his perspective). So each of them gets at least half the cake, based on their own judgment!

How can *three* people do a similar “cut-and-choose” procedure, so that each person gets at least $1/3$ of the cake, from their own perspective?

Solution: A divides the cake into 3 equal pieces, from her perspective. B chooses the 2 largest pieces as she sees it, and equalizes them (by cutting a sliver off one and adding it to the other). From these, C takes the largest, as he sees it. If C takes the piece B didn't equalize, then A takes the piece + sliver and B takes the other equalized piece. If C takes one of the equalized pieces, then B takes the other equalized piece and A gets the piece B thought was smallest.

2. Each pirate ship is going to send representatives to serve in the pirates' governing body, the Assembly of Rovers, Raiders, and Rogues (ARRR). There are **10 seats** in the Assembly, and each of the 4 ships should get a number of seats proportional to its crew size. Here are the ships and crew sizes:

Ship	Crew Size
Queen Anne's Revenge	60
Jolly Roger	45
Black Pearl	30
Good Ship Lollipop	15

How many seats should each ship get?

Solution: In this case, there is a perfectly proportional solution: Queen Anne's Revenge gets 4, Jolly Roger gets 3, Black Pearl gets 2, and Good Ship Lollipop gets 1.

3. A few years later, the pirate ships' crew sizes have changed, and they agree to recalculate how many seats each ship will receive.

Ship	Crew Size
Queen Anne's Revenge	77
Jolly Roger	46
Black Pearl	30
Good Ship Lollipop	27

Now how would you divide up the 10 seats among the ships, *and why?*

Solution: Answers may vary between 4-3-2-1, 5-3-1-1, 4-2-2-2 (any others?).

Several of the ships have offered competing proposals for apportioning seats. All of the proposals calculate a *divisor* $D = (\text{total \# of pirates})/(\text{\# of seats})$. (So in this case $D = 180/10 = 18$.) Then, for each ship, that ship's *quota* Q of seats is computed as $Q = \text{crew size}/D$. (So for instance Queen Anne's Revenge has $Q = 77/18 \approx 4.28$.)

Proposal 1 (Captain Jack Sparrow, Black Pearl):

1. For each ship, **round its quota Q down** and assign the ship that many seats to start with. Call the leftover fractional part of Q LEFTOVER. (For instance, if $Q = 4.6$, that ship gets 4 seats to start and that ship's value of LEFTOVER is .6)
2. If there are still seats left to be assigned, give them to the ships with the **largest LEFTOVER values**.

Proposal 2 (Blackbeard, Queen Anne's Revenge):

1. For each ship, **round its quota Q down** and assign the ship that many seats to start with.
2. If there are still seats to be assigned, **decrease D** as necessary and repeat step 1 until D is small enough that exactly S seats are assigned.

Proposal 3 (Shirley Temple, Good Ship Lollipop):

1. For each ship, **round its quota Q up** and assign the ship that many seats to start with.
2. If more than S seats have been assigned, **increase D** as necessary and repeat step 1 until D is large enough that exactly S seats are assigned.

Proposal 4 (Captain Hook, Jolly Roger):

1. For each ship, **round its quota Q to the nearest whole number** and assign the ship that many seats to start with.
2. If there are still seats remaining, **decrease D** as necessary and repeat step 1 until D is small enough that exactly S seats are assigned. If on the other hand, more than S seats have been assigned, **increase D** as necessary and repeat step 1 until D is large enough that exactly S seats are assigned.

For each ship, calculate its quota and then figure out how many seats it would get under each plan. You can use the following table to record your answers:

Ship	Crew Size	Quota	Seats under...			
			Proposal 1	Proposal 2	Proposal 3	Proposal 4
Queen Anne's Revenge	77	4.28	4	5	4	4
Jolly Roger	46	2.56	3	3	2	3
Black Pearl	30	1.67	2	1	2	2
Good Ship Lollipop	27	1.5	1	1	2	1

Solution: See filled-out table. For Proposal 2, the modified divisor can be from 15.1 to 15.3. For Proposal 3, anywhere from 23.0 to 25.6. For Proposal 4, 18.1 to 18.4.

In the U.S. House of Representatives, each state gets a number of representatives proportional to its population. For instance, California is the largest state with about 37,000,000 people, so we get the most seats (53). Wyoming, the smallest state with only 550,000 people, gets the fewest representatives (only 1).

4. (a) As mentioned above, Wyoming has only 1 seat (and therefore 1 congressional district) for its 550,000 people. Montana also has only 1 seat (and 1 district) with a population of about 970,000. What is the average number of people for each district in California? (Use 2 digits of accuracy, such as “about 37,000,000” rather than “37,249,105.”)
Solution: About 700,000.
 - (b) The current U.S. population is about 310,000,000. There are 435 members of the House of Representatives. On average, how many people are in each Congressional district in the U.S.? (Use 2 digits of accuracy.) **Solution:** About 710,000.
 - (c) Residents of Washington, D.C. don’t get a representative (since D.C. is not a state). Guess how many people live there? **Solution:** About 600,000 who live in the city itself have no representative. (About 5.4 million people live in the larger D.C. Metro Area, which includes parts of Maryland and northern Virginia.)
 - (d) According to the first Census, the United States in 1790 had about 4,000,000 people (and only 13 states!). At this time, the number of seats in the House was 65. On average, how large was each district (2 digits of accuracy)? **Solution:** About 62,000.
5. The number of Representatives per state is recalculated every 10 years. Based on the 1990 census, Oklahoma had 6 House seats. According to the 2000 census, Oklahoma’s population grew by almost 10 percent between 1990 and 2000 (from 3.15 million to 3.45 million). But Oklahoma’s number of seats went down, from 6 to 5. What is the most likely explanation?
Solution: On average, other states’ population grew by *more* than 10% during this period.
 6. Every weekend, Cinderella and her stepsisters go to balls. Some weekends there are two balls scheduled at the same time, and the stepsisters often argue about which one to go to. Cinderella’s stepmother has finally gotten so sick of it, she tells them whenever any of the sisters can’t decide which ball to go to, the 3 of them must vote on what to do. But she adds, “Cinderella, since you are not as smart (or as pretty!) as Lucinda and Florinda, your vote only counts for 1, and their votes count for 5 each.” Should Cinderella complain? **Solution:** She doesn’t need to bother arguing, because she’s in the same situation she would have been in if they each had 1 vote! (Namely, any 2 out of the 3 of them will carry a motion—Lucinda and Florinda with 10 votes out of 11, and Cinderella + one stepsister with 6 votes out of 11.)
 7. After some time, the pirate ships’ crews have changed, and there are two new ships. They decide to try a new system for ARRR: Each ship will have only one actual representative, but that person will get a number of *votes* proportional to their ship’s crew size, where a majority of votes is necessary to carry any motion.

Here are the ships, along with the number of votes the pirate who represents that ship will get:

District	Number of Votes
Queen Anne's Revenge	9
Jolly Roger	9
Black Pearl	7
The Good Ship Lollipop	3
Tarantula	1
Fancy	1

It so happens that these numbers are perfectly proportional to the respective crew sizes. Nevertheless, some upstarts on The Good Ship Lollipop, the Tarantula, and the Fancy feel that the system is unfair, and they've hired you as their lawyer to sue the other ships. Make an argument that the system is unfair to these ships (even though they have a proportional number of votes!).

Solution: The way the numbers work out, once you consider the votes cast by the Queen Anne's Revenge, the Jolly Roger, and the Black Pearl, the outcome of the vote is always determined. (For instance, even if Queen Anne's Revenge and Black Pearl vote yes way, and Jolly Roger votes no, so far the tally is 16-9; so regardless of which way the 5 votes from Lollipop, Tarantula, and Fancy go, the outcome will be yes.) Therefore, it's fair to say that The Good Ship Lollipop, Tarantula, and Fancy have *no power* in this voting system—they might as well have *zero* votes each!.