16 Adding and subtracting binary numbers

Materials for the lesson and homework: a couple of regular pencils, an eraser, a pencil sharpener; a binary abacus. It is not hard to make one out of the standard decimal abacus by separating two left-most beads on every wire from the remaining eight beads with a floral band or shoelace tied around the wires.

Warm-up

Problem 16.1

- *You have five locks and five keys. Each key works with only one lock. How many trials do you need in the worst case to match each lock to its key?*

- *You have ten locks and ten keys. Each key works with only one lock. How many trials do you need in the worst case to match each lock to its key?*

16.1 Lesson

One way to add and subtract binary numbers is to convert them to the decimal form, perform the needed operation, and convert the result back to the binary.

Example 16.1 *To subtract* B 11011 *from* B 100101*, let us first convert them to decimals using the table below.*

measured weight	*basic weights*						
	64	**32**	**16**	**8**	**4**	**2**	**1**
		1	0	0	1	0	1
			1	1	0	1	1

According to the table, $B\ 100101 = 32 + 4 + 1 = 37$ *and* $B\ 11011 = 16 + 8 + 2 + 1 = 27$*. Therefore,* $B\ 100101 - B\ 11011 = 37 - 27 = 10 = 8 + 2 = B\ 1010$.

Problem 16.2 *Use the method of example 16.1 to add* B 1011 *and* B 1101*. We have broken down the computation into steps for you.*

- *Find the decimal value of* B 1011*.*

$$B\ 1011 = \underline{\hspace{6cm}}$$

- *Find the decimal value of* $B\ 1101$.

$$B\ 1101 = \underline{\hspace{6cm}}$$

- *Add up the resulting decimal numbers.*

$$Decimal\ sum = \underline{\hspace{6cm}}$$

- *Represent the result as a sum of basic binary numbers.*

$$\underline{\hspace{1.5cm}} = \underline{\hspace{6cm}}$$

- *Convert the answer to the binary form.*

$$Answer = B\ \underline{\hspace{5cm}}$$

Problem 16.3 *Use the method of example 16.1 to subtract* $B\ 10$ *from* $B\ 1100$. *We have broken down the computation into steps for you.*

- *Find the decimal value of* $B\ 10$.

$$B\ 10 = \underline{\hspace{6cm}}$$

- *Find the decimal value of* $B\ 1100$.

$$B\ 1100 = \underline{\hspace{6cm}}$$

- *Subtract the decimal value of* $B\ 10$ *from the decimal value of* $B\ 1100$.

$$Difference = \underline{\hspace{6cm}}$$

- *Represent the difference as a sum of basic binary numbers.*

_______ = ______________________________

- *Convert the answer to the binary form.*

$Answer = B$ ______________________

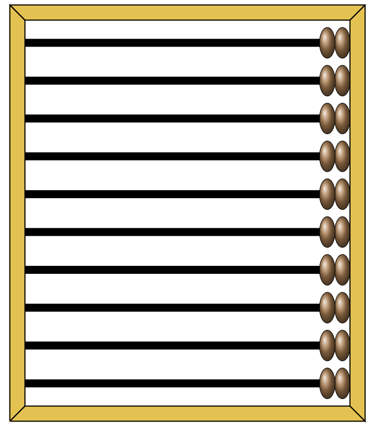
Problem 16.4 *Show that* $B\ 10 + B\ 1 = B\ 11$*.*

It is much faster to add and subtract binary numbers without switching to the decimal form. We will do it in two different ways in parallel, pen-on-paper and using a binary abacus. Our main tool for pen-on-paper computations will be the binary addition table.

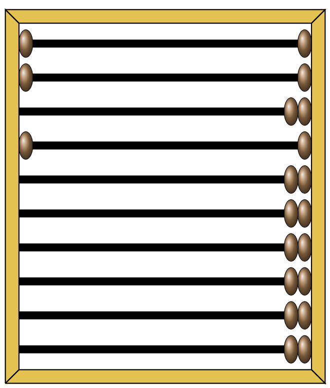
+	0	1
0	0	1
1	1	10

A binary abacus is a device similar to the decimal abacus, but having two beads on each wire. The value of a bead on the first wire is 1, the value of a bid on the second wire is 2, the value of a bid on the third wire is 4, etc. Please see the picture below.

B $1 = 1$
B $10 = 2$
B $100 = 4$
B $1,000 = 8$
B $10,000 = 16$
B $100,000 = 32$
B $1,000,000 = 64$
B $10,000,000 = 128$
B $100,000,000 = 256$
B $1,000,000,000 = 512$

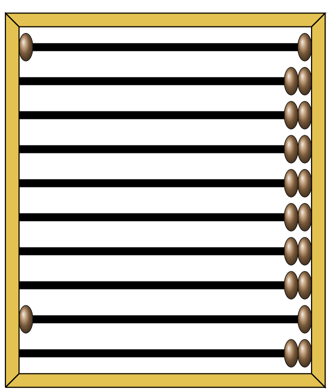


Problem 16.5 *Write down the binary numbers shown on the abaci below in the top rows to the right of the corresponding abaci. Write down the decimal values of the numbers in the second rows on the right. Use blanc spaces for computations if needed.*



*B*____________

*Decimal:*_____



*B*____________

*Decimal:*_____

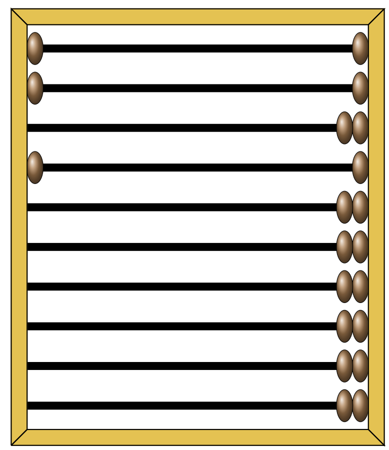
Let us redo problem 16.2 using a pen-on-paper method, called *long addition*. To make computations more visual, we will use a binary abacus in parallel.

In problem 16.2, we added B 1011 and 1101. We will not write the letter B in what follows to shorten notations.

• Setup: rewrite the numbers one under another so that their digits are aligned. The digit representing the basic binary number 1 for the first number must be written under the digit representing the basic binary number 1 for the second number; the digit representing the basic binary number 2 for the first number must be written under the digit representing the basic binary number 2 for the second number; and so forth. Put a line underneath.

$$\begin{array}{r} 1\ 0\ 1\ 1 \\ +\quad 1\ 1\ 0\ 1 \\ \hline \end{array}$$

• Setup: make the first number on the abacus.



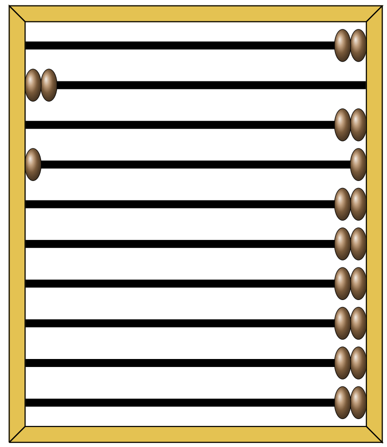
• Add the right-most single-digit binary number using the table.

$$1 + 1 = 10$$

The number 0 goes into the right-most column, the number 1 carries over to the next one.

$$\begin{array}{rcccc} & & & 1 & \\ & 1 & 0 & 1 & 1 \\ + & 1 & 1 & 0 & 1 \\ \hline & & & & 0 \end{array}$$

• Add 1 to the number on the abacus. Since $1 + 1 = 10$, we end up with no beads on the first wire and two beads on the second wire.



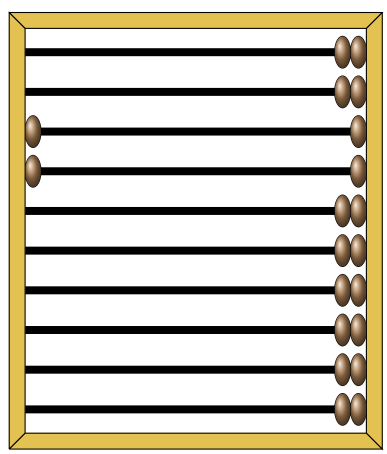
• Now we have to add three single-digit binary numbers in the second column from the right.

$$1 + 1 + 0 = 10$$

The number 0 stays in the column, the number 1 carries over to the next one.

$$\begin{array}{rcccc} & & 1 & & \\ & 1 & 0 & 1 & 1 \\ + & 1 & 1 & 0 & 1 \\ \hline & & & 0 & 0 \end{array}$$

• Let us replace two beads on the second wire by their equivalent, one bead on the third wire.



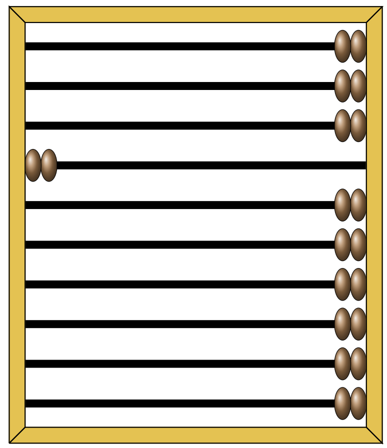
• Let us now add three single-digit binary numbers in the third column from the right.

$$1 + 0 + 1 = 10$$

The number 0 stays in the column, the number 1 carries over to the next one.

$$\begin{array}{rcccc} & 1 & & & \\ & 1 & 0 & 1 & 1 \\ + & 1 & 1 & 0 & 1 \\ \hline & & 0 & 0 & 0 \end{array}$$

• We add one more third wire bead to the one we already have. We replace two beads on the third wire by one bead on the fourth wire.



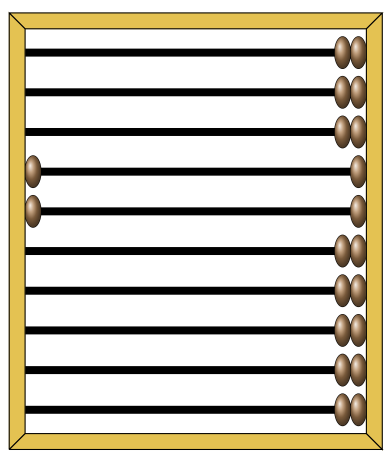
• Finally, we have to add three single-digit binary numbers in the left-most column.

$$1 + 1 + 1 = 10 + 1 = 11$$

The digit 1 from the right of the number 11 stays in the column, the other 1 carries over.

$$\begin{array}{rccccc} & 1 & & & & \\ & & 1 & 0 & 1 & 1 \\ + & & 1 & 1 & 0 & 1 \\ \hline & 1 & 1 & 0 & 0 & 0 \end{array}$$

• Tow beads on the fourth wire are equivalent to one bead on the fifth wire. Plus, we have to add 1 more to the fourth wire.



We are done: B 1011 + B 1101 = B 11000.

Problem 16.6 *Use long addition to add the following binary numbers. Use a binary abacus to check your work.*

$$\begin{array}{r} 1\ 1 \\ +\quad 1 \\ \hline \end{array} \qquad \begin{array}{r} 1\ 1 \\ +\ 1\ 1 \\ \hline \end{array}$$

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1 \\ +\quad 1\ 1\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 1\ 1\ 1\ 0\ 1 \\ +\ 1\ 0\ 0\ 1\ 0 \\ \hline \end{array}$$

In the following problem, we double-check the last part of problem 16.6 by re-doing the computation in the decimal form.

Problem 16.7

- *Find the decimal value of B* 11101.

B 11101 = ________________________

- *Find the decimal value of B* 10010.

B 10010 = ________________________

- *Add up the resulting decimal numbers.*

Decimal sum = ____________________

- *Represent the result as a sum of basic binary numbers.*

______ = _______________________

- *Convert the answer to the binary form.*

Answer = *B* __________________

- *Was your solution to the last part of problem 16.6 correct? Circle the correct answer.*

Yes *No*

It's time to take a look at some terminology related to subtraction. The number we subtract is called a *subtrahend*. The number we subtract from is called a *minuend*. The result is called the *difference*. In the number sentence $5 - 3 = 2$ for example, 5 is a minuend, 3 is a subtrahend, and 2 is the difference. The terminology is derived from Latin. The English word *subtraction* originates from the Latin *subtrahere*. The latter has two roots: *sub* means either *under* or *from under* while *trahere* means to *pull*. So, *subtraction* literally means *pulling from under*. The word *minuere* means to *reduce* or *diminish*. This way, a *minuend* is something that is diminished.

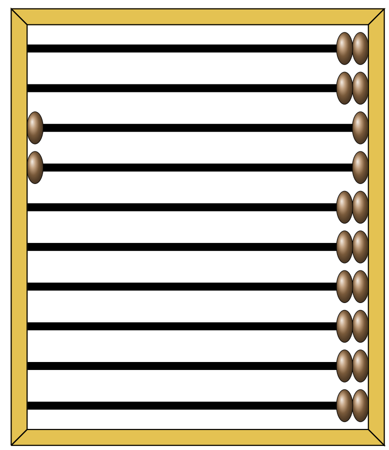
You will find the root *sub* in the words *submarine*, *subtropical*, and others. The word *trahere* has evolved over the centuries into the English word *tract*, as in *subtraction*. Another important example is a *tractor*, a machine that pulls.

Recall that in problem 16.3, we subtracted B 10 from B 1100. Let us use two different ways in parallel to solve the problem without switching to the decimals. The first is a pen-on-paper way, called *long subtraction*. The second way employs the binary abacus.

• Setup: let us align the numbers, the subtrahend below the minuend, and let us draw a line underneath. It is very important that the digit representing the basic binary number 1 in the subtrahend is written under the digit representing the basic binary number 1 in the minuend; the digit representing the basic binary number 2 in the subtrahend is written under the digit representing the basic binary number 2 in the minuend; and so forth.

$$\begin{array}{r} 1\ 1\ 0\ 0 \\ -\quad\quad 1\ 0 \\ \hline \end{array}$$

• Setup: let us make the minuend on the abacus.



• Similar to long addition, we start from the right-most digit: $0 - 0 = 0$.

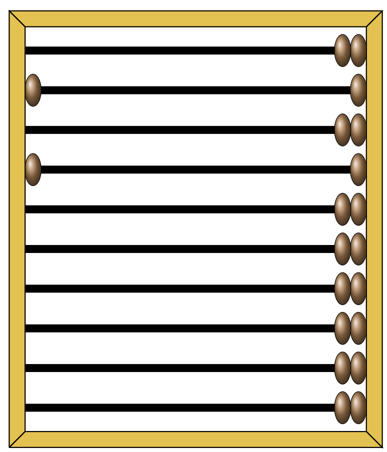
$$\begin{array}{rrrrr} & 1 & 1 & 0 & 0 \\ - & & & 1 & 0 \\ \hline & & & & 0 \end{array}$$

• Nothing changes on the abacus for the same reason: $0 - 0 = 0$.

• In the next column to the left, we need to subtract 1 from 0. To do so, let us borrow from the column further left. The dot atop the third column 1 shows the borrowing. A basic binary number is twice the previous basic binary number. Since $2 =_B 10$, the borrowed 1 shows as the binary 10 in the previous column. Since $1 + 1 = 10$ in the binary system, $10 - 1 = 1$. We write the latter 1 down in the second column. Since we have borrowed 1 from the minuend's third column, there is nothing left there, so we write 0 underneath.

$$\begin{array}{rrrrr} & & & 10 & \\ & 1 & \dot{1} & 0 & 0 \\ - & & & 1 & 0 \\ \hline & & 0 & 1 & 0 \end{array}$$

• We need to subtract 1 on the second wire, but there are no beads available there. Borrowing from the third wire leaves no beads on it, but makes two beads on the second wire. Now we take away one of the two.



• The minuend has 1 in the left-most column, the subtrahend has nothing. The missing digit means a zero, so $1 - 0 = 1$. Writing down this 1 in the left-most position of the difference finishes the computation.

• There are no more digits in the subtrahend, so nothing changes on the abacus.

$$\begin{array}{rcccc} & & & 10 & \\ & 1 & \dot{1} & 0 & 0 \\ - & & & 1 & 0 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

We are done: B 1100 − B 10 = B 1010.

Problem 16.8 *Use long subtraction to find the following differences of binary numbers. Use a binary abacus to check your work.*

$$\begin{array}{rccc} & 1 & 1 & 0 \\ - & & & 1 \\ \hline \end{array} \qquad \begin{array}{rccc} & 1 & 0 & 1 \\ - & & 1 & 1 \\ \hline \end{array}$$

$$\begin{array}{rccccc} & 1 & 1 & 0 & 1 & 1 \\ - & & & 1 & 0 & 1 \\ \hline \end{array} \qquad \begin{array}{rccccc} & 1 & 1 & 1 & 0 & 1 \\ - & 1 & 0 & 1 & 1 & 0 \\ \hline \end{array}$$

Problem 16.9 *Fill the blanks in the following binary addition problem.*

$$
\begin{array}{rcccc}
 & & 1 & 1 & 0 \\
+ & & & \underline{} & 1 \\
\hline
 & 1 & \underline{} & 0 & \underline{}
\end{array}
$$

Problem 16.10 *Fill the blanks in the following binary subtraction problem.*

$$
\begin{array}{rccc}
 & 1 & \underline{} & \underline{} \\
- & & 1 & \underline{} \\
\hline
 & & & 1
\end{array}
$$

Problem 16.11 *Is it possible to arrange twelve chairs into six rows with three chairs in each row? If you think it, show how. If you think it isn't, explain why.*

16.2 Homework

Finish solving all the problems from class. Explain to your parents the meaning of the Latin words *sub* and *trahere*. Show them how the roots are used in the word *subtraction*.

Problem 16.12

- *Give an example of an English word having the root* <u>*sub*</u> *different from the words "subtraction" and "subtropical" used in the lesson.*

- *What do the words "to tract" mean?*

- *Give an example of an English word having the root* <u>*tract*</u> *different from the words "subtraction" and "tractor" used in the lesson.*

Problem 16.13 *Use long addition to add the following binary numbers. Use a binary abacus to check your work.*

$$\begin{array}{r} 1\ 0\ 1 \\ +\quad 1\ 1 \\ \hline \end{array} \qquad \begin{array}{r} 1\ 0\ 1\ 1 \\ +\quad 1\ 1\ 1 \\ \hline \end{array}$$

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1 \\ +\quad 1\ 0\ 1\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 1\ 0\ 1\ 1\ 1 \\ +\ 1\ 1\ 0\ 1\ 0 \\ \hline \end{array}$$

Problem 16.14 *Use long subtraction to find the following differences of binary numbers. Use a binary abacus to check your work.*

$$\begin{array}{r} 1\ 0\ 0 \\ -\quad\ 1 \\ \hline \end{array} \qquad \begin{array}{r} 1\ 0\ 0\ 0 \\ -\quad 1\ 0\ 1 \\ \hline \end{array}$$

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \\ -\quad 1\ 1\ 1\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 1\ 0\ 1\ 0\ 1\ 0 \\ -\quad 1\ 0\ 1\ 1\ 1 \\ \hline \end{array}$$

Problem 16.15 *Fill the blanks in the following binary addition problem.*

$$\begin{array}{rcccccc} & & 1 & 1 & \sqcup & 0 & 1 \\ + & & & \sqcup & 1 & \sqcup & \sqcup \\ \hline & \sqcup & \sqcup & \sqcup & 1 & 1 & 0 \end{array}$$

Problem 16.16 *Fill the blanks in the following binary subtraction problem.*

$$
\begin{array}{rccccc}
 & 1 & 0 & 0 & 1 & 0 \\
- & & 1 & \sqcup & \sqcup & \sqcup \\
\hline
 & & & & 1 & \sqcup
\end{array}
$$

Problem 16.17 *Show that given any three integers, you can always find two such that their difference is even.*