A Game of Stones: An Introduction to Nim

Jeremy Brightbill
Los Angeles Math Circle

Take-Away Games, Part 1

The game of 21: There are 21 stones in a pile. Players take 1, 2 or 3 turns away on their turn. The person to take the last stone wins!
1) Play one or two games of 21 with your partner.
2) Suppose the pile only has 1 stone. Who has the advantage, the first or second player? What if the pile has two stones? Three stones? Four? Five? Six?
3) Suppose the pile has 10 stones. Who has the advantage?
4) Suppose the pile has 12 stones. Who has the advantage?
5) Which player has the advantage in 21? Test out your theory by playing a couple of games with your partner. Try going both first and second and see the difference!

Take-Away Games, Part 2

1) How does the winning strategy change if the goal is **not** to take the last stone?
2) How does the winning strategy change if only 1 or 2 stones can be removed?
3) How does the winning strategy change if 2, 3, or 4 stones can be removed? (If only one stone remains, you still take it.)

SOS

1) Play a few games of SOS with your partner. Try some short games (length 3, 4, 5), and some longer games (length 12, 13, 14).
2) Does either player have a winning strategy for a game of length 3?
3) Play a game of length 4. Suppose the first player places an S in the left-hand column:
S _ _ _
What is a really good move for Player 2 to make?
4) Suppose S _ _ S appears somewhere in the row. Is it possible for there to be a draw?
5) Play games of length 6, 7, 8, 9. At the beginning, work together to put an S _ _ S somewhere in the row, then play normally. Which player wins?
6) Can you find a winning strategy for a game of length 7? Which player wins?
7) Does this winning strategy work for a game of length 8? What about length 9? Length 10?
8) Play a game of length 16 with your partner. Can you find a winning strategy? Which player wins?
9) Which player would win a game of length 40? 41?

Nim, Part 1

1) With your partner, play a few games of Nim in simple configurations and determine which ones are winning and losing positions. Calculate the Nim number of each position. Here are a few good examples to try:
a) Rows of 2 and 2.
b) Rows of 2 and 3.
c) Two rows of the same length.
d) Two rows of different length.
e) Rows of 1, 2, and 1.
f) Rows of 1, 2, and 2.
g) Rows of 1, 2, and 3.
2) What do all the losing positions have in common? Based on this, what should your strategy be?
3) Test your conjecture by playing a few winning and losing games of Nim with your partner.

Extra Problems

1) Suppose the rows of Nim are of length 1, 2, 4, 8, $\cdots$, 2^n. For which values of n is this a winning configuration?
2) Suppose a game of Nim has 5 rows. What is the smallest possible losing game?
3) How does the winning strategy change if the goal is to **not** take the final piece?

Another Game

Rules for "Chomp": Consider a rectangular array of stones (arranged like the squares in a candy bar). Players 1 and 2 take turns removing stones. However, each time a stone is removed, all stones below or to the right are also removed. (Each person is taking a rectangular "bite" out of the candy bar, hence the name.) The person who moves last loses (i.e., the person who takes the poisonous upper-left corner piece).

1) Play a few games of "Chomp" with your partner.
2) No matter the board size, the first player in "Chomp" always has a winning strategy. Can you prove why? (Hint: The answer is very short and simple, but requires some cleverness. Suppose the second player claims to have a winning strategy. How can the first player turn this strategy to his advantage?)
3) Though a winning strategy exists, the winning strategy for $m \times n$ Chomp still has not been discovered! Can you be the first to find it?