Fractal Dimension

An amalgam of excerpts of works by
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The main question we aim to address in this handout is:

> How can we tell the dimension of a set? For example, certain objects are one-dimensional (e.g. line segments in 3D space), or two dimensional (e.g. surfaces in 3D) or three-dimensional. How can this be detected?

1 Minkowski content.

Let X be a subset of $2D$ space, and let's assume that X is "bounded" (i.e., it fits inside a disk of a sufficiently large radius).

We'll denote by $N_t(X)$ the t-neighborhood of X, i.e. $N_t(X)$ is the set of all points which are distance at most t from some point of X.

If X is a single point, $N_t(X)$ is a disk of radius t centered at that point. If X consists of two points, then N_t consists of the two disks each of radius t centered at those two points.

Problem 1 In each of the following cases: (a) X a single point; (b) X a line segment; (c) X a disk, find a formula for the area of $N_t(X)$.

The idea is to now look at the rate at which the area shrinks as t decreases to zero.

Problem 2 (a) Assume that $f(t) = t^d$. Find an expression for

$$\frac{\log f(t)}{\log t}.$$

(b) Assume now that $f(t) = Ct^d$ for some fixed C. Show that for small t,

$$\frac{\log f(t)}{\log t}$$

is very close to d.

(c) Guess a formula for the dimension of X in terms of the areas of $N_t(X)$.

This formula is called the *Minkowski dimension* of X.

Problem 3 Redo everything in this section for a subset X' of $3D$ space. Explain why if X' is contained in a plane, then its dimension (measured using areas in that plane) is the same as the one measured using volumes in three dimensions. Now redo everything for a subset X'' of 1-dimensional space.

2 Packing and covering dimensions.

Assume now that X is a subset of the line and that X is bounded. Denote by $K_t(X)$ the smallest number of intervals each of length t needed to cover X. Denote by $P_t(X)$ the largest number of intervals of length t so that the intersections of these intervals with X are disjoint and nonempty.

Problem 4 Let X be (a) a point; (b) a line segment. Find $K_t(X)$.

Problem 5 Show the following:

$$tP_t(X) \leq \text{Area of } N_t(X) \leq tK_t(X).$$

Problem 6 Show that $P_t(X) \geq K_{2t}(X)$. *Hint.* Assume that X is covered by p intervals of length t, and the number p is minimal, i.e., $p = P_t(X)$. Now replace each interval with another interval of twice the length, but centered at the same point. Show that these intervals must cover X (if they don't, was p really minimal?)

Problem 7 We thus have (for $t < 1$, so that $\log t < 0$):

$$\frac{\log tP_t(X)}{|\log t|} \leq \frac{\log\ \text{Area of } N_t(X)}{|\log t|} \leq \frac{\log tK_t(X)}{|\log t|} \leq \frac{\log tP_{t/2}(X)}{|\log t|}.$$

Replace t by $2s$ in the last equation and notice that

$$\frac{\log 2sP_s(X)}{|\log 2s|} = \frac{\log sP_s(X) + \log 2}{|\log s + \log 2|} \approx \frac{\log sP_s(X)}{|\log s|}$$

for very small s. Conclude that for t extremely small, all of the numbers

$$\frac{\log tP_t(X)}{|\log t|}, \quad \frac{\log tK_t(X)}{|\log t|}, \quad \frac{\log \text{Area of } N_t(X)}{|\log t|}$$

are approximately the same. How does the last formula compare to the one you found in the previous section?

Definition 1. If for very small t, the quantity

$$\frac{\log tK_t(X)}{|\log t|} + 1 = \frac{\log K_t(X)}{|\log t|} \quad (\text{for } t < 1),$$

becomes approximately equal to a number d, we call d the covering (or packing) dimension of X. We note that the "+1" has been added to the quantities we considered above in order to make, for example, the dimension of a line segment equal 1.

Problem 8 What is the covering dimension of the Cantor set C_∞?

Hint: Estimate the covering and packing numbers of C_∞. Now compute:

$$\frac{\log K_t}{|\log t|}$$

This is the dimension of the Cantor set.

3 More Exercises on Fractal Dimensions

Above, we defined $K_t(X)$ and $P_t(X)$ to be the least number of intervals of length t required to cover a set X in the real line, and the largest number of intervals that could be backed disjointly into X respectively. From these, we defined a *covering* or *packing* dimension. Now we will use higher dimensional shapes to cover and pack X in two or three dimensions. If the shape is Y, $K_m(X)$ will be the least number of copies of Y, scaled to have measure (length, area, or volume) m, required to cover X, and $P_m(X)$ will be the largest number of non-overlapping copies of Y, scaled to have measure m, that can be placed with non-empty intersection with X.

The packing/covering dimension of a subset of n-dimensional space is then defined by

$$\frac{n \log P_m(X)}{|\log(m)|} \approx \frac{n \log K_m(X)}{|\log(m)|}$$

for very small m. This turns out not to depend very much on the choice of shape Y, but we will not prove that here.

Problem 9 The Koch snowflake is the curve created by following this recursive process for infinitely many steps:

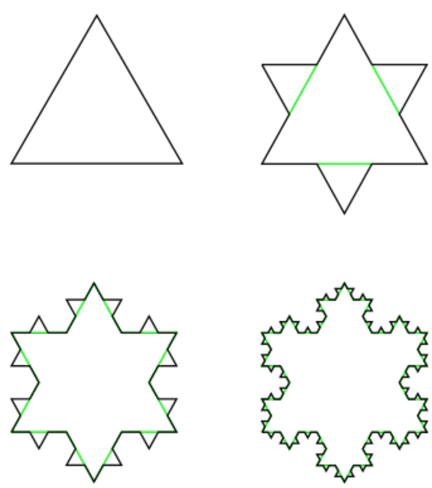
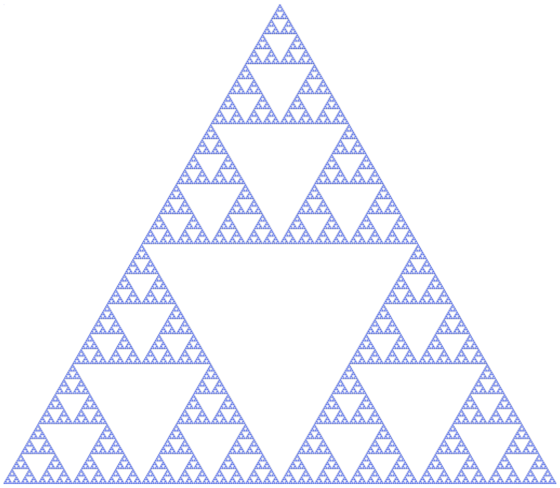


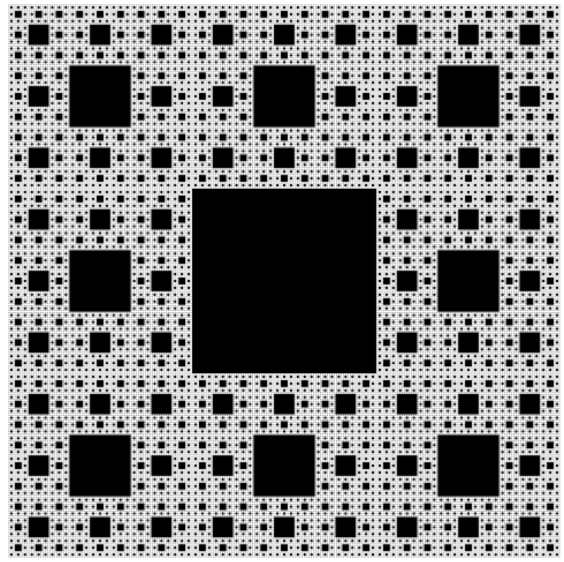
Figure 1: Koch Snowflake `https://en.wikipedia.org/wiki/File:KochFlake.svg`

Compute the area of the interior of this curve, and the length of the nth stage in the construction of the curve. Using an equilateral triangle for Y, compute the covering and packing dimensions of the curve (just the boundary).

Problem 10 Compute the covering and packing dimensions of the Sierpinski triangle and Sierpinski carpet with equilateral triangles and squares respectively:



(a) Sierpinski Triangle `https://commons.wikimedia.org/wiki/File:Sierpinski_triangle.svg`



(b) Sierpinski Carpet (The **white region**, not the black) `https://en.wikipedia.org/wiki/File:Sierpinski_carpet_6.svg`

Problem 11 Now jumping to 3 dimensions, try using cubes to compute the covering and packing dimensions of the Menger sponge, which is constructed by iterating this process:

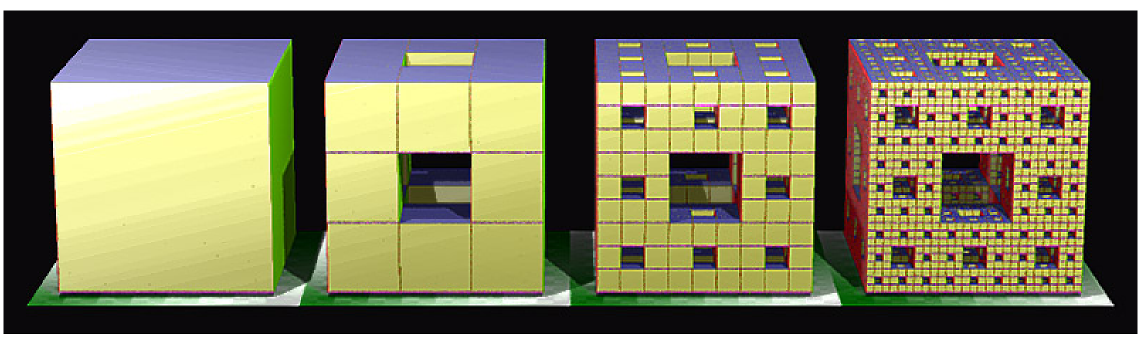


Figure 3: Menger Sponge `https://commons.wikimedia.org/wiki/File:Menger_sponge_(Level_0-3).jpg`